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ONLINE APPENDIX FOR "LOW INTEREST RATES, MARKET POWER, AND PRODUCTIVITY GROWTH"H

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APPENDIX A: PROOFS

A.1. *Proof of claims in Sections 3 and 4*

Proof of Proposition 1 The solution to HJB equations (1) through (3) imply that equilibrium investment and value functions must satisfy $\eta_s = v_{s+1} - v_s$ for $s \in \{-1, 0, 1\}$. The HJB equations can thus be re-written as

$$(r + \eta_s/2 + \eta_{-s})\, v_s = \pi_s + \eta_s v_{s+1}/2 + \eta_{-s} v_{s-1} \qquad \text{for } s \in \{-1, 0, 1\}. \tag{A.1}$$

Substitute using $v_2 = \pi_2/r$, $v_{-2} = \pi_{-2}/r$, $v_1 = v_2 - \eta_1$, $v_0 = v_2 - \eta_1 - \eta_0$, and $v_{-1} = v_2 - \eta_1 - \eta_0 - \eta_{-1}$, the HJB equations become a system of 3 quadratic equations involving 3 endogenous variables $\{\eta_{-1}, \eta_0, \eta_1\}$ with exogenous parameters $\{\pi_s\}$ and r. That $d\eta_s/dr < 0$ follows from totally differentiating the system of equations and applying the implicit function theorem.

We prove a generalized version of the limiting result that as $r \to 0$, $\eta_1 \to \infty$, $\eta_{-1} \to \infty$, and $(\eta_1 - \eta_{-1}) \to \infty$, under a quadratic cost function with a leader disadvantage. Specifically, define $c_s = 1$ if $s < 1$ and $c_s = c$ if $s = 1$, and write the HJB equation for state $s \in \{-1, 0, 1\}$:

$$r v_s = \max_{\eta} \pi_s - c_s \eta^2/2 + \eta\,(v_{s+1} - v_s) + \eta_{-s}\,(v_{s-1} - v_s).$$

The parameter c is a cost shifter for the leader. The example in Section 3 has $c = 1$. When $c > 1$, leader holds a cost disadvantage relative to the follower. We now prove the limiting result for a generic c. Optimal investment satisfies $\eta_{-1} = v_0 - v_{-1}$, $\eta_0 = v_1 - v_0$, and $c\eta_1 = v_2 - v_1$. After substituting these expressions into the HJB equation and then taking the limit $r \to 0$, we obtain

$$v_1 \sim \frac{\eta_1 v_2 + 2\eta_{-1} v_0}{\eta_1 + 2\eta_{-1}}, \qquad v_0 \sim \frac{v_1 + 2v_{-1}}{3}, \qquad v_{-1} \sim \frac{\eta_{-1} v_0 + 2\eta_1 v_{-2}}{\eta_{-1} + 2\eta_1},$$

where we use $x \sim y$ to denote $\lim_{r\to 0}(x - y) = 0$. Using optimal investment decisions to substitute out v_{-1}, v_0 and v_1, we obtain

$$c\eta_1 \sim \frac{8\eta_{-1}\,(v_2 - v_{-2})}{6\eta_1 + 9\eta_{-1}}, \qquad \eta_{-1} \sim \frac{2\eta_1\,(v_2 - v_{-2})}{6\eta_1 + 9\eta_{-1}},$$

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thereby implying $c\eta_1^2 \sim 4\eta_{-1}^2$. As $r \to 0$, $v_2 - v_{-2} \to \infty$, implying that $\eta_1 \to \infty$, $\eta_{-1} \to \infty$, and $(\eta_1 - \eta_{-1}) \to \infty$ if and only if $c < 4$. In particular, when the leader does not have a cost disadvantage ($c = 1$), the difference between leader and follower investment diverges.

Proof of Lemmas 4.1 and 4.2 The CES demand within each market implies that the market share of firm i is $\delta_i \equiv \frac{p_i y_i}{p_1 y_1 + p_2 y_2} = \frac{p_i^{1-\sigma}}{p_1^{1-\sigma} + p_2^{1-\sigma}}$. Under Bertrand competition, the price of a firm with productivity z_i must solve $p_i = \frac{\sigma(1-\delta_i)+\delta_i}{(\sigma-1)(1-\delta_i)} \lambda^{-z_i}$, with markup $m_i \equiv \frac{p_i}{\lambda^{-z_i}} = \frac{\sigma(1-\delta_i)+\delta_i}{(\sigma-1)(1-\delta_i)}$ and profits $\pi_i = \delta_i \left(\frac{p_i - \lambda^{-z_i}}{p_i}\right)$. Now define ρ_s as the relative price between leader and follower in a market with productivity gap s. Taking ratios of the prices and re-arrange, we derive that ρ_s must solve $\rho_s^\sigma = \lambda^{-s} \frac{\left(\sigma \rho_s^{\sigma-1} + 1\right)}{\sigma + \rho_s^{\sigma-1}}$. Market share is therefore $\delta_s = \frac{\rho_s^{1-\sigma}}{\rho_s^{1-\sigma} + 1}$ for the leader and $\delta_{-s} = \frac{1}{\rho_s^{1-\sigma} + 1}$ for the follower and profits are $\pi_s = \frac{1}{\sigma \rho_s^{\sigma-1} + 1}$ and $\pi_{-s} = \frac{\rho_s^{\sigma-1}}{\sigma + \rho_s^{\sigma-1}}$, respectively. Leader's markup is $m_s = \frac{\sigma + \rho_s^{1-\sigma}}{\sigma - 1}$ and follower's markup is $m_{-s} = \frac{\sigma \rho_s^{1-\sigma} + 1}{(\sigma-1)\rho_s^{1-\sigma}}$.

The fact that follower's flow profits are convex in s follows from algebra. Moreover, $\lim_{s\to\infty} \rho_s^\sigma \lambda^s = 1/\sigma$; hence, for large s, $\pi_s \approx \frac{1}{\sigma^{\frac{1}{\sigma}} \lambda^{-\frac{\sigma-1}{\sigma}s} + 1}$ and $\pi_{-s} \approx \frac{1}{\sigma^{\frac{2\sigma-1}{\sigma}} \lambda^{\frac{\sigma-1}{\sigma}s} + 1}$. The eventual concavity of π_s and $(\pi_s + \pi_{-s})$ as $s \to \infty$ is immediate. Also note that, as $s \to \infty$, $\pi_s \to 1$, $\pi_{-s} \to 0$, $m_s \to \infty$, $m_{-s} \to 0$.

Proof of Lemma 4.5 The expression $g = \ln \lambda \left(\sum_{s=0}^{\infty} \mu_s \eta_s + \mu_0 \eta_0\right)$ shows that aggregate growth is equal to $\ln \lambda$ times the weighted-average investment rate of firms at the frontier—leaders and neck-and-neck firms. In a steady-state, the growth rate of the productivity frontier must be the same as the growth rate of followers; hence, aggregate growth rate g can also be written as $g = \ln \lambda \left(\sum_{s=1}^{\infty} \mu_s \left(\eta_{-s} + \kappa\right)\right)$.

To prove the expression formally, we proceed in two steps. First, we express aggregate productivity growth as a weighted average of productivity growth in each market. We then use the fact that, given homothetic within-market demand, if a follower in state s improves productivity by one step (i.e. by a factor λ) and a leader in state $s-1$ improves also by one step, the net effect is equivalent to one step improvement in the overall productivity of a single market.

Let $p(\nu) \equiv \left[p_1(\nu)^{1-\sigma} + p_2(\nu)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$ be the price index of a single market ν. We can equivalently index for markets not using ν but instead using (s, z^F), the productivity gap and the productivity of the follower. The growth rate g of aggregate productivity defined in (12) is equal to $-\frac{d \ln P}{dt}$, where P is the ideal consumer price index, and can be written as:

$$g \equiv \frac{d \ln \lambda^Z}{dt} = -\frac{d \ln P}{dt} = -\frac{d \int_0^1 \ln p(\nu)\, d\nu}{dt} = -\sum_{s=0}^{\infty} \mu_s \times \frac{d \left[\int_{z^F} \ln p\left(s, z^F\right) dF\left(z^F\right)\right]}{dt}.$$

Now recognize that productivity growth rate in each market, $-\frac{d \ln p\left(s, z^F\right)}{d \ln t}$, is a function of only the productivity gap s and is invariant to the productivity of follower, z^F. Specifically, suppose the follower in market (s, z^F) experiences an innovation, the market price index becomes $p\left(s-1, z^F+1\right)$. If instead the leader experiences an innovation, the price index becomes $p\left(s+1, z^F\right)$. The corresponding log-changes in price indices are respectively

$$a_s^F \equiv \ln p\left(s-1, z^F+1\right) - \ln p\left(s, z^F\right) = -\ln \lambda + \ln\left[\rho_{s-1}^{1-\sigma} + 1\right]^{\frac{1}{1-\sigma}} - \ln\left[\rho_s^{1-\sigma} + 1\right]^{\frac{1}{1-\sigma}},$$

$$a_s^L \equiv \ln p\left(s+1, z^F\right) - \ln p\left(s, z^F\right) = \ln\left[\rho_{s+1}^{1-\sigma} + 1\right]^{\frac{1}{1-\sigma}} - \ln\left[\rho_s^{1-\sigma} + 1\right]^{\frac{1}{1-\sigma}},$$

where ρ_s is the implicit function defined in the proof for Lemma 1. The log-change in price index is independent of z^F in either case. Hence, over time interval $[t, t+\Delta]$, the change in price index for markets with state variable s at time t follows

$$\Delta \ln p\left(s, z^F\right) = \begin{cases} a_s^L & \text{with probability } \eta_s \Delta, \\ a_s^F & \text{with probability } \left(\eta_{-s} + \kappa \cdot \mathbf{1}\left(s \neq 0\right)\right)\Delta. \end{cases}$$

The aggregate productivity growth can therefore be written as

$$g = -\mu_0 2\eta_0 a_0 - \sum_{s=1}^{\infty} \mu_s \times \left(\eta_s a_s^L + \left(\eta_{-s} + \kappa\right) a_s^F\right),$$

where $a_0 \equiv a_0^F = a_0^L$. Finally, note if both leader and follower in a market experiences productivity improvements, regardless of the order in which these events happen, the price index in the market changes by a factor of λ^{-1}: $a_s^F + a_{s-1}^L = a_s^L + a_{s+1}^F = -\ln\lambda$ for all $s \geq 1$. Hence,

$$\begin{aligned} g &= -\mu_0 2\eta_0 a_0 - \sum_{s=1}^{\infty} \mu_s \times \left(\eta_s a_s^L + \left(\eta_{-s} + \kappa\right) a_s^F\right) \\ &= -\mu_0 2\eta_0 a_0 - \sum_{s=1}^{\infty} \mu_s \times \left(\eta_s a_s^L + \left(\eta_{-s} + \kappa\right)\left(-\ln\lambda - a_{s-1}^L\right)\right) \\ &= \ln\lambda \cdot \sum_{s=1}^{\infty} \mu_s \left(\eta_{-s} + \kappa\right) - \left(\sum_{s=1}^{\infty} \mu_s \times \left(\eta_s a_s^L - a_{s-1}^L\left(\eta_{-s} + \kappa\right)\right) + \mu_0 2\eta_0 a_0\right). \end{aligned}$$

Given that steady-state distribution $\{\mu_s\}$ must follow equations (10) and (11), we know

$$\begin{aligned} &\sum_{s=1}^{\infty} \mu_s \times \left(\eta_s a_s^L - a_{s-1}^L\left(\eta_{-s} + \kappa\right)\right) + \mu_0 2\eta_0 a_0 = \\ &\sum_{s=1}^{\infty} \mu_s \eta_s a_s^L + \mu_0 2\eta_0 a_0 - \left(\sum_{s=1}^{\infty} \mu_s a_{s-1}^L \left(\eta_{-s} + \kappa\right)\right) = 0 \end{aligned}$$

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Hence aggregate growth rate simplifies to $g = \ln\lambda \cdot \sum_{s=1}^{\infty} \mu_s\left(\eta_{-s} + \kappa\right)$, which traces the growth rate of productivity laggards. We can also apply equations (10) and (11) again to express productivity growth as a weighted average of frontier growth: $g = \ln\lambda \cdot \left(\sum_{s=1}^{\infty} \mu_s \eta_s + 2\mu_0\eta_0\right)$.

A.2. *An example with leapfrogging*

"No-sudden-leapfrog" is the source of strategic asymmetry in the paper. We extend the example in Section 3 to allow for partial leapfrogging show a strict form of "no-leapfrog" is not necessary for strategic asymmetry. Suppose upon a successful innovation, the follower can jump from state -1 to state 2 with probability p, and to state 0 with probability $1-p$. The HJB equations can be written as

$$rv_1 = \max_{\eta} \pi_1 - \eta^2/2 + \eta\left(v_2 - v_1\right) + \eta_{-1}\left(\left(1-p\right)v_0 + pv_{-2} - v_1\right) \tag{A.2}$$

$$rv_0 = \max_{\eta} \pi_0 - \eta^2/2 + \eta\left(v_1 - v_0\right) + \eta_0\left(v_{-1} - v_0\right) \tag{A.3}$$

$$rv_{-1} = \max_{\eta} \pi_{-1} - \eta^2/2 + \eta\left[(1-p)\left(v_0 - v_{-1}\right) + p\left(v_2 - v_{-1}\right)\right] + \eta_1\left(v_{-2} - v_{-1}\right) \tag{A.4}$$

The HJB equations reflect the fact that with probability p, follower's innovation raises follower's value to v_2 and knocks leader's value down to v_{-2}. The model in Section 3 corresponds to the case where $p = 0$.

Using the same proof strategy as for Proposition 1, one can show that as $r \to 0$, equilibrium investment η_1 and η_{-1} both diverge to infinity, and their ratio, $x \equiv \eta_1/\eta_{-1}$, must satisfy the following cubic equation:

$$(1+2p)\,x^3 + 2p\,(2+p)\,x^2 - 2\,(2+p)\,x - p\,(1+2p) = 0.$$

For all $p < 1$, the positive solution to the cubic equation always features $x > 1$, implying that $(\eta_1 - \eta_{-1}) \to \infty$. That is, unless the follower always leapfrogs with probability $p = 1$, the leader-follower strategy asymmetry that we highlight is always present, and the leader always responds to low interest rates more than the follower.

A.3. *Proof of claims in Sections 5.2 and 5.3*

Section 5 maintains the assumption that investment cost is linear, $c(\eta_s) = c \cdot \eta_s$ for $\eta_s \in [0, \eta]$. As discussed in Section 4.2, we assume the investment space is sufficiently large—$c\eta > \pi_\infty$ and $\eta > \kappa$—so that firms can compete intensely if they choose to—and c is not prohibitively high relative to the gains from becoming a leader ($c\kappa < \pi_\infty - \pi_0$)—otherwise no firm has any incentive to ever invest.

Proof of Lemma 5.1 Recall $n+1 \equiv \min\{s | s \geq 0, \eta_s < \eta\}$ is the first state in which market leaders choose not to invest, and $k+1 \equiv \min\{s | s \leq 0, \eta_s < \eta\}$ is the first state in which followers choose not to invest. Suppose $n < k$, i.e. leader invests in states 1 through n whereas follower invests in states 1 through at least $n+1$. We first show that, if these investment decisions were optimal, the value functions of both leader and follower in state $n+1$ must be supported by certain lower bounds. We then reach for a contradiction, showing that, if $n < k$, then market power is too transient to support these lower bounds on value functions.

The HJB equation for the leader in state $n+2$ implies

$$\begin{aligned} rv_{n+2} &= \max_{\eta_{n+2} \in [0,\eta]} \pi_{n+2} + \eta_{n+2}\left(v_{n+3} - v_{n+2} - c\right) + \left(\eta_{-(n+2)} + \kappa\right)\left(v_{n+1} - v_{n+2}\right) \\ &\geq \pi_{n+2} + (\eta + \kappa)\left(v_{n+1} - v_{n+2}\right). \end{aligned} \tag{A.5}$$

That leader does not invest in $s = n+1$ implies $c \geq v_{n+2} - v_{n+1}$; combining with (A.5) to get

$$rv_{n+1} \geq \pi_{n+2} - c\,(\eta + \kappa + r)\,.$$

The HJB equation for the follower in state $n+1$ implies

$$\begin{aligned} rv_{-(n+1)} &= \max_{\eta_{-(n+1)} \in [0,\eta]} \pi_{-(n+1)} + \left(\eta_{-(n+1)} + \kappa\right)\left(v_{-n} - v_{-(n+1)}\right) - c\eta_{-(n+1)} \\ &\geq \pi_{-(n+1)} + \kappa\left(v_{-n} - v_{-(n+1)}\right). \end{aligned} \tag{A.6}$$

That follower invests in $s = n+1$ implies $c \le v_{-n} - v_{-(n+1)}$; combining with (A.6) to get

$$rv_{-(n+1)} \ge \pi_{-(n+1)} + c\kappa. \tag{A.7}$$

Combining this with the earlier inequality involving rv_{n+1}, we obtain an inequality on the joint value $w_{n+1} \equiv v_{n+1} + v_{-(n+1)}$:

$$rw_{n+1} \ge \pi_{n+2} + \pi_{-(n+1)} - c(\eta + r) \tag{A.8}$$

We now show that inequalities (A.7) and (A.8) cannot both be true. To do so, we construct alternative economic environments with value functions $\hat{w}_1^{(0)}$ and $\hat{v}_{-1}^{(0)}$ such that $\hat{w}_1^{(0)} \ge w_{n+1}$ and $\hat{v}_{-1}^{(0)} \ge v_{-(n+1)}$; we then show that even these dominating value functions $\hat{w}_1^{(0)}$ and $\hat{v}_{-1}^{(0)}$ cannot satisfy both inequalities.

First, fix n and fix investment strategies (leader invests until state $n+1$ and follower invests at least through $n+1$); suppose for all states $1 \le s \le n+1$, follower's profits are equal to $\pi_{-(n+1)}$ and leader's profits are equal to π_{n+2}; two firms each earn $\frac{\pi_{-(n+1)}+\pi_{n+2}}{2}$ in state zero. The joint profits in this modified economic environment are independent of the state by construction; moreover, the joint flow profits always weakly dominate those in the original environment and strictly dominate in state zero ($\pi_{n+2} + \pi_{-(n+1)} \ge \pi_1 + \pi_{-1} > 2\pi_0$). Let $\hat{w}_s$ denote the value function in the modified environment; $\hat{w}_s > w_s$ for all $s \le n+1$.

Consider the joint value in this modified environment but under alternative investment strategies. Let $\bar{n}$ index for investment strategies: leader invests in states 1 through $\bar{n}$ whereas the follower invests at least through $\bar{n}+1$. Let $\hat{w}_s^{(\bar{n})}$ denote the joint value in state s under investments indexed by $\bar{n}$. We argue that $\hat{w}_{\bar{n}+1}^{(\bar{n})}$ is decreasing in $\bar{n}$. To see this, note the joint flow payoffs in all states 0 through $\bar{n}$ is constant by construction and is equal to $x \equiv (\pi_{n+2} + \pi_{-(n+1)} - 2c\eta)$—total profits net of investment costs—and the joint flow payoff in state $\bar{n}+1$ is $(\pi_{n+2} + \pi_{-(n+1)} - c\eta) = x + c\eta$. $\hat{w}_{\bar{n}+1}^{(\bar{n})}$ is equal to a weighted average of x/r and $(x + c\eta)/r$, and the weight on $(x+c\eta)/r$ is higher when $\bar{n}$ is smaller. Hence, $\hat{w}_{\bar{n}+1}^{(\bar{n})}$ is decreasing in $\bar{n}$, and that $\hat{w}_1^{(0)} \ge \hat{w}_{n+1}^{(n)} > w_{n+1}$. The same logic also implies $\hat{v}_0^{(0)} = \frac{1}{2}\hat{w}_0^{(0)} > \frac{1}{2}w_0 = v_0$.

Consider follower's value $\hat{v}_{-1}^{(0)}$ in the alternative environment, when investment strategies are indexed by zero, i.e. firms invest in states 0 and -1 only. We know $\hat{v}_{-1}^{(0)}$ must be higher than $v_{-(n+1)}$ because

$$\hat{v}_{-1}^{(0)} = \frac{\pi_{-(n+1)} - c\eta + \kappa\hat{v}_0^{(0)}}{r+\kappa+\eta} > \frac{\pi_{-(n+1)} - c\eta + \kappa v_0}{r+\kappa+\eta} \ge \frac{\pi_{-(n+1)} - c\eta + \kappa v_{-n}}{r+\kappa+\eta} = v_{-(n+1)}.$$

We now show that inequalities $r\hat{v}_{-1}^{(0)} \ge \pi_{-(n+1)} + c\kappa$ and $r\hat{w}_1^{(0)} \ge \pi_{n+2} + \pi_{-(n+1)} - c(\eta + r)$ cannot both hold. We can explicitly solve for the value functions from the HJB equations:

$$\hat{w}_0^{(0)} = \frac{\pi_{n+2} + \pi_{-(n+1)} - 2c\eta + 2\eta\hat{w}_1^{(0)}}{r+2\eta}$$

$$\hat{w}_1^{(0)} = \frac{\pi_{n+2} + \pi_{-(n+1)} - c\eta + (\eta+\kappa)\hat{w}_0^{(0)}}{r+\eta+\kappa}$$

$$\hat{v}_{-1}^{(0)} = \frac{\pi_{-(n+1)} - c\eta + (\eta+\kappa)\hat{w}_0^{(0)}/2}{r+\eta+\kappa}$$

Solving for $\hat{w}_1^{(0)}$ and $\hat{v}_{-1}^{(0)}$, we obtain

$$r\hat{w}_1^{(0)} = \pi_{n+2} + \pi_{-(n+1)} - c\eta\left(1 + \frac{\eta+\kappa}{r+3\eta+\kappa}\right)$$

$$(r+\eta+\kappa)\, r\hat{v}_{-1}^{(0)} = r\left(\pi_{-(n+1)} - c\eta\right) + (\eta+\kappa)\left(\frac{\pi_{n+2}+\pi_{-(n+1)}}{2} - c\eta\frac{r+2\eta+\kappa}{r+3\eta+\kappa}\right)$$

That $r\hat{v}_{-1}^{(0)} \geq \pi_{-(n+1)} + c\kappa$ implies

$$\begin{aligned}(r+\eta+\kappa)\, r\hat{v}_{-1}^{(0)} &= r\left(\pi_{-(n+1)} - c\eta\right) + (\eta+\kappa)\left(\frac{\pi_{n+2}+\pi_{-(n+1)}}{2} - c\eta\frac{r+2\eta+\kappa}{r+3\eta+\kappa}\right)\\ &\geq (r+\eta+\kappa)\left(\pi_{-(n+1)} + c\kappa\right)\end{aligned}$$

$$\implies (\eta+\kappa)\left(\frac{\pi_{n+2}-\pi_{-(n+1)}}{2} - c\eta\frac{r+2\eta+\kappa}{r+3\eta+\kappa}\right) \geq (r+\eta+\kappa)\, c\kappa + c\eta r$$

Since $\frac{\pi_{n+2}-\pi_{-(n+1)}}{2} \leq \frac{\pi_{n+2}}{2} < c\eta$, it must be the case that

$$(\eta+\kappa)\, c\eta > (r+\eta+\kappa)\, c\kappa + c\eta r + (\eta+\kappa)\, c\eta\frac{r+2\eta+\kappa}{r+3\eta+\kappa}.$$

On the other hand, that $r\hat{w}_1^{(0)} \geq \pi_{n+2} + \pi_{-(n+1)} - c(\eta+r)$ implies $r \geq \eta\frac{\eta+\kappa}{r+3\eta+\kappa}$; hence the previous inequality implies

$$\begin{aligned}(\eta+\kappa)\, c\eta &> (r+\eta+\kappa)\, c\kappa + (\eta+\kappa)\, c\eta\frac{\eta}{r+3\eta+\kappa} + (\eta+\kappa)\, c\eta\frac{r+2\eta+\kappa}{r+3\eta+\kappa}\\ &= (r+\eta+\kappa)\, c\kappa + (\eta+\kappa)\, c\eta,\end{aligned}$$

which is impossible; hence $n \geq k$.

We now show that the follower does not invest in states $s \in \{k+1, ..., n+1\}$. First, note

$$\begin{aligned}(r+\eta+\kappa)\left(v_{-s} - v_{-s-1}\right) &= \pi_{-s} - \pi_{-s-1} + \kappa\left(v_{-s+1} - v_{-s}\right) + \eta\left(v_{-s-1} - v_{-s-2}\right)\\ &\quad + \max\left\{\eta\left(v_{-s+1} - v_{-s} - c\right), 0\right\} - \max\left\{\eta\left(v_{-s} - v_{-s-1} - c\right), 0\right\}.\end{aligned}$$

Suppose $v_{-s+1} - v_{-s} \geq (v_{-s} - v_{-s-1})$, then

$$(r+\eta+\kappa)\left(v_{-s} - v_{-s-1}\right) \geq \pi_{-s} - \pi_{-s-1} + \kappa\left(v_{-s+1} - v_{-s}\right) + \eta\left(v_{-s-1} - v_{-s-2}\right)$$

$$\implies (r+\eta)\left(v_{-s} - v_{-s-1}\right) \geq \pi_{-s} - \pi_{-s-1} + \eta\left(v_{-s-1} - v_{-s-2}\right).$$

If $v_{-s+1} - v_{-s} < (v_{-s} - v_{-s-1})$, then

$$\begin{aligned}(r+\eta)\left(v_{-s} - v_{-s-1}\right) &< \pi_{-s} - \pi_{-s-1} + \eta\left(v_{-s-1} - v_{-s-2}\right)\\ &\quad + \max\left\{\eta\left(v_{-s+1} - v_{-s} - c\right), 0\right\} - \max\left\{\eta\left(v_{-s} - v_{-s-1} - c\right), 0\right\}\\ &\leq \pi_{-s} - \pi_{-s-1} + \eta\left(v_{-s-1} - v_{-s-2}\right).\end{aligned}$$

To summarize, for all s,

$$v_{-s+1}-v_{-s}\geq\left(v_{-s}-v_{-s-1}\right)\iff\left(r+\eta\right)\left(v_{-s}-v_{-s-1}\right)\geq\pi_{-s}-\pi_{-s-1}+\eta\left(v_{-s-1}-v_{-s-2}\right) \tag{A.9}$$

Now suppose $\eta_{-k-1}=0$ but $\eta_{-s'}=\eta$ for some $s'\in\{k+2,...,n+1\}$. This implies

$$v_{-(k-1)}-v_{-k}\geq c>v_{-k}-v_{-k-1}<v_{-s'+1}-v_{-s'},$$

implying there must be at least one $s\in\{k+2,...,n+1\}$ such that $v_{-s+1}-v_{-s}\geq v_{-s}-v_{-s-1}<v_{-s-1}-v_{s-2}$. Applying (A.9),

$$\left(r+\eta\right)\left(v_{-s}-v_{-s-1}\right)\geq\pi_{-s}-\pi_{-s-1}+\eta\left(v_{-s-1}-v_{-s-2}\right) \tag{A.10}$$

$$\left(r+\eta\right)\left(v_{-s-1}-v_{-s-2}\right)<\pi_{-s-1}-\pi_{-s-2}+\eta\left(v_{-s-2}-v_{-s-3}\right) \tag{A.11}$$

Inequality (A.10) and $v_{-s}-v_{-s-1}<v_{-s-1}-v_{-s-2}$ implies $r\left(v_{-s}-v_{-s-1}\right)>\pi_{-s}-\pi_{-s-1}$; convexity in follower's profit functions further implies $r\left(v_{-s}-v_{-s-1}\right)>\pi_{-s-1}-\pi_{-s-2}$. Substitute into inequality (A.11), and using the fact $v_{-s}-v_{-s-1}<v_{-s-1}-v_{s-2}$, we deduce it must be the case that $\left(v_{-s-2}-v_{s-3}\right)>\left(v_{-s-1}-v_{-s-2}\right)$. Applying (A.9) again,

$$\left(r+\eta\right)\left(v_{-s-2}-v_{-s-3}\right)<\pi_{-s-2}-\pi_{-s-3}+\eta\left(v_{-s-3}-v_{-s-4}\right).$$

That $r\left(v_{-s-2}-v_{-s-3}\right)>\pi_{-s-2}-\pi_{-s-3}$ further implies $\left(v_{-s-3}-v_{-s-4}\right)>\left(v_{-s-2}-v_{-s-3}\right)$. By induction, we can show $v_{s-1}-v_{s-2}<v_{s-2}-v_{s-3}<\cdots<v_{-n}-v_{-(n+1)}$. But

$$\left(r+\eta+\kappa\right)\left(v_{-n}-v_{-(n+1)}\right)\leq\pi_{-n}-\pi_{-(n+1)}+\kappa\left(v_{-n+1}-v_{-n}\right)+\eta\left(v_{-n+1}-v_{-n+1}\right)$$

$$\implies\left(r+\eta\right)\left(v_{-n}-v_{-(n+1)}\right)\leq\pi_{-n}-\pi_{-(n+1)}$$

which is a contradiction, given convexity of the profit functions. Hence, we have shown $v_{-k}-v_{-(k+1)}\geq v_{-s}-v_{-s-1}$ for all $s\in\{k+1,...,n+1\}$, establishing that follower cannot invest in these states.

Proof of Lemma 5.2 Given the cutoffs (n,k), aggregate productivity growth is (from Lemma 4.5) $g=\ln\lambda\cdot\left(\sum_{s=1}^{n}\mu_s\eta+2\mu_0\eta\right)$. The steady-state distribution must follow

$$\mu_s\eta=\begin{cases}\mu_1\left(\eta+\kappa\right)/2 & \text{if } s=0\\ \mu_{s+1}\left(\eta+\kappa\right) & \text{if } 1\leq s\leq k-1\\ \mu_{s+1}\kappa & \text{if } k\leq s\leq n+1\\ 0 & \text{if } s>n+1\end{cases} \tag{A.12}$$

Hence we can rewrite the aggregate growth rate as

$$\begin{aligned}g&=\ln\lambda\cdot\left(2\mu_0\eta+\sum_{s=1}^{k-1}\mu_s\eta+\sum_{s=k-1}^{n}\mu_s\eta\right)\\&=\ln\lambda\cdot\left(\mu_1\left(\eta+\kappa\right)+\sum_{s=2}^{k}\mu_s\left(\eta+\kappa\right)+\sum_{s=k}^{n+1}\mu_s\kappa\right)\\&=\ln\lambda\cdot\left(\mu^C\left(\eta+\kappa\right)+\mu^M\kappa\right),\end{aligned}$$

as desired. To solve for μ_0, μ^C, and μ^M as functions of n and k, we use (A.12) to write μ_s as a function of μ_{n+1} for all s. Let $\alpha \equiv \kappa/\eta$, then

$$\mu_s = \begin{cases} \mu_{n+1}\alpha^{n+1-s} & \text{if } n+1 \geq s \geq k \\ \mu_{n+1}\alpha^{n+1-k}\left(1+\alpha\right)^{k-s} & \text{if } k-1 \geq s \geq 1 \\ \mu_{n+1}\alpha^{n+1-k}\left(1+\alpha\right)^{k}/2 & \text{if } s=0 \end{cases}$$

Hence $\mu_0 = \mu_{n+1}\alpha^{n+1-k}\left(1+\alpha\right)^k/2$. The share of markets in the competitive and monopolistic regions can be written, respectively, as

$$\mu^M = \mu_{n+1}\sum_{s=k+1}^{n+1}\alpha^{n+1-s} = \mu_{n+1}\frac{1-\alpha^{n-k+1}}{1-\alpha},$$

$$\mu^C = \mu_{n+1}\alpha^{n+1-k}\sum_{s=1}^{k}\left(1+\alpha\right)^{k-s} = \mu_{n+1}\alpha^{n-k}\left(\left(1+\alpha\right)^k - 1\right).$$

Proof of Lemma 5.3 Given $k \geq 1$, the share of markets in the competitive region is

$$\begin{aligned} \mu^C &= \sum_{s=1}^{k}\mu_s = \mu_1 + \underbrace{\mu_1\left(1+\alpha\right)^{-1}}_{=\mu_2} + \cdots + \underbrace{\mu_1\left(1+\alpha\right)^{-(k-1)}}_{=\mu_k} \\ &= \underbrace{\mu_0\frac{\kappa+\eta}{2\eta}}_{=\mu_1}\frac{1-\left(1+\alpha\right)^{-k}}{1-\left(1+\alpha\right)^{-1}} \geq \mu_0\frac{\kappa+\eta}{2\eta} \end{aligned}$$

Aggregate growth rate can be re-written as

$$g = \ln\lambda\cdot\left[\left(1-\mu_0\right)\kappa + \mu^C\eta\right] \geq \ln\lambda\cdot\left[\left(1-\mu_0\right)\kappa + \mu_0\frac{\kappa+\eta}{2}\right] \geq \ln\lambda\cdot\kappa.$$

Aggregate investment is $I = 2\eta\left(\mu^C+\mu_0\right) + \eta\left(\mu^M - \mu_{n+1}\right)$. In a steady-state, it must be that $2\eta\mu_0 + \eta\left(\mu^M - \mu_{n+1}\right) = \left(\eta+\kappa\right)\mu^C + \kappa\mu^M$, thus $I = 2\eta\mu^C + \kappa\left(1-\mu_0\right) \geq \kappa$, as desired.

A.4. *Proof of claims in Section 5.4*

Consider the following recursive equations of value functions $\{u_s\}_{s=-\infty}^{\infty}$:

$$ru_{s+1} = \lambda_{s+1} + p_{s+1}\left(u_s - u_{s+1}\right) + q\left(u_{s+2} - u_{s+1}\right) \tag{A.13}$$

where λ_{s+1} is the flow payoff, p_{s+1} and q are respectively the Poisson rate of transition from state $s+1$ into state s and state $s+2$. Given u_s and $\Delta u_s \equiv u_{s+1} - u_s$, we can solve for all u_{s+t}, $t > 0$ as recursive functions of u_s and Δu_s. The recursive formulation generically does not have a closed-form representation. However, as $r \to 0$, the value functions do admit asymptotic closed form expressions, as Proposition A.1 shows. In what follows, let $\sim$ denote asymptotic equivalence as $r \to 0$, i.e. $x \sim y$ iff $\lim_{r\to 0}\left(x - y\right) = 0$.

Proposition A.1. Consider value functions $\{u_s\}_{s=-\infty}^{\infty}$ satisfying (A.13). Fix state s and integer $t>0$. Suppose $\lambda_{s'}\equiv\lambda$ and $p_{s'}\equiv p$ for all states $s\le s'\le t$. Let $\delta\equiv\frac{ru_s-\lambda}{q}$, $a\equiv\frac{p}{q}$, $b\equiv\frac{r}{q}$, then for all $t>0$,

$$\begin{aligned} u_{s+t}-u_s\sim&\left(\Delta u_s\right)\frac{1-a^t}{1-a}+\delta\frac{t-\dfrac{a-a^t}{1-a}}{1-a}\\ &+\Delta u_s\cdot b\frac{(t-1)\left(1+a^t\right)(1-a)-(2-a)\left(a^t-a\right)}{(1-a)^3}\\ &+\delta b\frac{1}{(1-a)^3}\left(\left(\frac{(t-2)(t-1)}{2}+2a\right)(1-a)-(t-3)a^t-a(2-a)(t-1)\right) \end{aligned} \tag{A.14}$$

$$\begin{aligned} u_{s+t}-u_{s+t-1}\sim&\,\Delta u_s a^{t-1}+\delta\frac{1-a^{t-1}}{1-a}+\Delta u_s b\frac{\left((t-1)\left(1+a^t\right)-(t-2)\left(1+a^{t-1}\right)\right)}{(1-a)^2}\\ &-\Delta u_s b\frac{\left((2-a)\left(a^t-a^{t-1}\right)\right)}{(1-a)^3}+\frac{\delta b}{(1-a)^2}\frac{(t-2)(t-1)-(t-2)(t-3)}{2}\\ &-\frac{\delta b}{(1-a)^3}(t-3)a^t+(t-4)a^{t-1}-a(2-a)\Bigg). \end{aligned} \tag{A.15}$$

If $t\to\infty$ as $r\to 0$, then the formulas can be simplified as follows:

1. If $a<1$, then $u_{s+t}-u_{s+t-1}\sim\Delta u_s a^{t-1}+\frac{\delta}{1-a}+\frac{b\Delta u_s}{(1-a)^2}$; further,
   (a) if $r\Delta u_s\to 0$, then $u_{s+t}-u_s\sim\Delta u_s\frac{1}{1-a}+\frac{t\delta}{1-a}$;
   (b) if $r\Delta u_s\not\to 0$, then $r\left(u_{s+t}-u_s\right)\sim\frac{r\Delta u_s}{1-a}$.
2. Suppose $a>1$ and $r\Delta u_s\to 0$.
   (a) If $\Delta u_s+\frac{\delta}{a-1}\not\sim 0$, then $r\left(u_{s+t}-u_s\right)\sim\left(\Delta u_s+\frac{\delta}{a-1}\right)\frac{ra^t}{a-1}$ and $r\left(u_{s+t}-u_{s+t-1}\right)\sim\left(\Delta u_s+\frac{\delta}{a-1}\right)ra^{t-1}$.
   (b) If $\Delta u_s+\frac{\delta}{a-1}\sim 0$, then $u_{s+t}-u_s\sim-\frac{b\delta}{(1-a)^4}\cdot a^{t+1}$.

Suppose $\lambda_{s'}$ and $p_{s'}$ are state-dependent. Let $\lambda\ge\lambda_{s'}$ and $p\le p_{s'}$ for all $s\le s'\le t$. The formulas in (A.14) and (A.15) provide asymptotic lower bounds for $u_{s+t}-u_{s+t-1}$ and $u_{s+t}-u_s$. Conversely, if $\lambda\le\lambda_{s'}$ and $p\ge p_{s'}$ for all $s\le s'\le t$, then the formulas provide asymptotic upper bounds for $u_{s+t}-u_{s+t-1}$ and $u_{s+t}-u_s$.

REMARK: *Proposition A.1. expresses u_{s+t} and Δu_{s+t} as functions of u_s and Δu_s. One can also apply the Proposition write u_s and Δu_s as functions of Δu_{s+t} and u_{s+t}. Proposition A.1. thus enables us to solve for value functions asymptotically, and we apply it repeated throughout the rest of this appendix.*

Proof of Proposition A.1. First suppose $\lambda_{s'}\equiv\lambda$ and $p_s\equiv p$ are constant for all states $s\le s'\le t$. Given u_s and Δu_s, we can solve for value functions u_{s+t} as

$$u_{s+1}-u_s=\Delta u_s$$

$$\begin{cases} u_{s+2}-u_{s+1}=a\Delta u_s+b\Delta u_s+\delta\\ u_{s+2}-u_s=(1+a)\Delta u_s+b\Delta u_s+\delta \end{cases} \tag{A.16}$$

$$\begin{cases} u_{s+3} - u_{s+2} = a^2 \Delta u_s + (1+2a)\, b \Delta u_s + (1+a)\,\delta + o\,(r) \\ u_{s+3} - u_s = (1+a+a^2)\,\Delta u_s + (1+1+2a)\, b\Delta u_s + (1+1+a)\,\delta + b\delta + o\,(r) \end{cases}$$

where $o\,(r)$ are terms that vanish as $r \to 0$. Applying the formula iteratively, we find

$$u_{s+t+1} - u_{s+t} = a^t \Delta u_s + \delta \sum_{z=0}^{t-1} a^z + b \Delta u_s \sum_{z=1}^{t} z a^{z-1} + b\delta \sum_{z=1}^{t-1} \sum_{m=1}^{z} m a^{m-1} + o\,(r)$$

$$u_{s+t+1} - u_s = \Delta u_s \sum_{z=0}^{t} a^z + \delta \sum_{z=0}^{t} \sum_{m=0}^{z-1} a^m + b \Delta u_s \sum_{z=1}^{t} \sum_{m=1}^{z} m a^{m-1} + b\delta \sum_{x=1}^{t-1} \sum_{z=1}^{x} \sum_{m=1}^{z} m a^{m-1} + o\,(r)$$

One obtains the proposition by applying the following formulas for power series summations:

1. $\sum_{z=0}^{t} a^z = \frac{1-a^{t+1}}{1-a}$;
2. $\sum_{z=0}^{t} \sum_{m=0}^{z-1} a^m = \frac{t+1-\frac{a-a^{t+1}}{1-a}}{1-a}$;
3. $\sum_{z=1}^{t} \sum_{m=1}^{z} m a^{m-1} = \frac{t\left(1+a^{t+1}\right)(1-a)-(2-a)\left(a^{t+1}-a\right)}{(1-a)^3}$;
4. $\sum_{x=1}^{t-1} \sum_{z=1}^{x} \sum_{m=1}^{z} m a^{m-1} = \frac{1}{(1-a)^3} \left(\frac{t(t-1)+4a}{2} (1-a) - (t-2)\, a^{t+1} - a\,(2-a)\, t \right)$.

Now suppose λ_s and p_s are state-dependent, and $\lambda \geq \lambda_{s'}$, $p \leq p_{s'}$ for all $s \leq s' \leq t$. Let $\delta_s \equiv \frac{ru_s - \lambda_s}{q}$, $a_s \equiv \frac{p_s}{q}$ and note $\delta_s > \delta \equiv \frac{ru_s - \lambda}{q}$, $a_s > a$. By re-writing equations in this proof as inequalities (e.g. rewrite (A.16) as $u_{s+2} - u_{s+1} > a\Delta u_s + b\Delta u_s + \delta$ and $u_{s+2} - u_s > (1+a)\,\Delta u_s + b\Delta u_s + \delta)$, the formulas in the Proposition provide asymptotic lower bounds for $u_{s+t} - u_{s+t-1}$ and $u_{s+t} - u_s$ as functions of u_s and Δu_s. Conversely, if $\lambda \leq \lambda_{s'}$ and $p \geq p_{s'}$ for all $s \leq s' \leq t$, then the formulas provide asymptotic upper bounds for $u_{s+t} - u_{s+t-1}$ and $u_{s+t} - u_s$. QED.

Proof of Lemma 5.4 Recall n and k are the last states in which the leader and the follower, respectively, chooses to invest in an equilibrium. Both n and k are functions of the interest rate r. Also recall that we use $w_s \equiv v_s + v_{-s}$ to denote the total firm value of a market in state s.

We first prove $\lim_{r\to 0} n = \infty$**.** Consider the sequence of value functions $\hat{v}_s$ generated by an alternative sequence investment decisions: leader follows equilibrium strategies and invests in n states whereas follower does not invest in any state. Under these alternative investments, flow payoff is higher in every state, hence the joint value of both firms is higher in every state—including state 0—thus $\hat{v}_0 \geq v_0$. One can further show by induction that the alternative value functions dominate the equilibrium value functions ($\hat{v}_s \geq v_s$) for all $s \geq 0$; intuitively, leader's value is higher in any state because it expects to spend more time in higher payoff states, since the follower does not invest. Also by induction one can show $\Delta v_s \geq \Delta \hat{v}_s$ for all $s \geq 0$; intuitively, when the follower does not invest, leader has less of an incentive to invest as well.

Now suppose n is bounded, and we look for a contradiction. Let N be the smallest integer such that (1) $N > n$ for all r, and (2) $\pi_N - \pi_0 > c\kappa$. Note $rv_N = r \cdot \frac{\pi_N + \kappa v_{N-1}}{r+\kappa} \to rv_{N-1}$ as $r \to 0$; hence $rv_N \sim rv_{N-1}$. By induction, because N is finite, $rv_s \sim rv_t \sim rv_{-s}$ for any $s, t \leq N$. Likewise, $r\hat{v}_s \sim r\hat{v}_t$ for any $s, t \leq N$. The fact that leader does not invest in state $N-1$ implies $\lim_{r\to 0} (v_N - v_{N-1}) < c \Longrightarrow \lim_{r\to 0} rv_{N-1} > \pi_N - c\kappa$, which further implies $\lim_{r\to 0} r\hat{v}_0 \geq \lim_{r\to 0} rv_0 = \lim_{r\to 0} rv_{N-1} > \pi_N - c\kappa$. Also note that $\Delta\hat{v}_0 > \Delta\hat{w}_0 = \frac{r\hat{w}_1 - (2\pi_0 - 2c\eta)}{r+2\eta} \to \frac{r\hat{w}_0 - (2\pi_0 - 2c\eta)}{2\eta} = \frac{r\hat{v}_0 - (\pi_0 - c\eta)}{\eta}$. We now put these pieces together and apply Proposition A.1 to compute a lower bound for $\Delta\hat{v}_n$ as a function of $\hat{v}_0$ and $\Delta\hat{v}_0$ (substitute

$u_s = \hat{v}_0$, $u_{s+t} = \hat{v}_N$, $a = \kappa/\eta$, $b = r/\eta$, $\delta = \frac{r\hat{v}_0 - (\pi_N - c\eta)}{\eta}$):

$$\begin{aligned}\lim_{r\to 0} \Delta\hat{v}_N &\geq \lim_{r\to 0}\left(\Delta\hat{v}_0\,(\kappa/\eta)^{N-1} + \frac{r\hat{v}_0 - (\pi_N - c\eta)}{\eta}\frac{1-(\kappa/\eta)^{N-1}}{1-\kappa/\eta}\right)\\ &> \lim_{r\to 0}\frac{r\hat{v}_0 - (\pi_0 - c\eta)}{\eta}(\kappa/\eta)^{N-1} + \frac{r\hat{v}_0 - (\pi_N - c\eta)}{\eta}\frac{1-(\kappa/\eta)^{N-1}}{1-\kappa/\eta}\\ &> \lim_{r\to 0}\frac{\pi_N - c\kappa - (\pi_0 - c\eta)}{\eta}(\kappa/\eta)^{N-1} + \frac{\pi_N - c\kappa - (\pi_N - c\eta)}{\eta}\frac{1-(\kappa/\eta)^{N-1}}{1-\kappa/\eta}\\ &> \lim_{r\to 0} c\,(\kappa/\eta)^{N-1} + \frac{c\,(\eta-\kappa)}{\eta}\frac{1-(\kappa/\eta)^{N-1}}{1-\kappa/\eta} \qquad = c\end{aligned}$$

where the last inequality follows the fact that $\pi_N - \pi_0 > c\kappa$. Thus $\lim_{r\to 0}\Delta v_N \geq \lim_{r\to 0}\Delta\hat{v}_N > c$ and the leader must invest in state N, a contradiction.

Next, suppose $\lim_{r\to 0} k = \infty$ but $(n-k)$ remain bounded. Let $\epsilon \equiv 2c\eta - \pi_\infty > 0$. The joint flow payoff $\pi_s + \pi_{-s} - 2c\eta$ is negative and bounded above by $-\epsilon$ in all states $s \leq k$. As $k \to \infty$, if $n-k$ remain bounded, then there are arbitrarily many states in which the total flow payoffs for both firms is negative and only finitely many states in which the flow payoffs may be positive. The firm value in state 0 is therefore negative. Since firms can always ensure non-negative payoffs by not taking any investment, this cannot be an equilibrium, reaching a contradiction. Hence $\lim_{r\to 0}(n-k) = \infty$.

To show $\lim_{r\to 0} k = \infty$, we first establish a few asymptotic properties of the model.

Lemma A.1. (1) $rv_n \sim \pi_\infty - c\kappa$; (2) $v_{n+1} - v_n \sim c$; (3) $r\,(n-k) \sim 0$; (4) $rk \sim 0$.

Proof. (1) The fact that leader invests in state n but not in state $n+1$ implies

$$\frac{\pi_{n+2} - rv_{n+1}}{r+\kappa} = v_{n+2} - v_{n+1} \leq c \leq v_{n+1} - v_n = \frac{\pi_{n+1} - rv_n}{r+\kappa}$$

$$\implies \pi_\infty - c\kappa = \lim_{r\to 0}(\pi_{n+2} - c\kappa) \geq \lim_{r\to 0} rv_n \geq \lim_{r\to 0}(\pi_{n+1} - c\kappa) = \pi_\infty - c\kappa, \text{ Q.E.D.}$$

(2) The claim follows from the previous one: $v_{n+1} - v_n = \frac{\pi_{n+1} - rv_n}{r+\kappa} \sim \frac{\pi_\infty - rv_n}{\kappa} \sim c$.

(3) The previous claims show $rv_n \sim \pi_\infty - c\kappa$ and $\Delta v_n \sim c$. We apply Proposition A.1 to iterate backwards and obtain a lower bound for $(v_k - v_n)$:

$$\lim_{r\to\infty} r\,(v_k - v_n) \geq \lim_{r\to\infty} -\frac{r^2}{\kappa^2}\frac{rv_n - (\pi_\infty - c\eta)}{(1-\eta/\kappa)^4}(\eta/\kappa)^{n-k+1} \sim -\frac{r^2}{\kappa^2}\frac{c\,(\eta-\kappa)}{(1-\eta/\kappa)^4}(\eta/\kappa)^{n-k+1}$$

Since $|\lim_{r\to 0} r\,(v_k - v_n)| \leq \pi_\infty$, $\lim_{r\to 0} r^2\,(\eta/\kappa)^{n-k+1}$ must remain bounded, implying $r\,(n-k) \sim 0$.

(4) We apply Proposition A.1 to find a lower bound for $w_k - w_0$ (where $a \equiv \eta/\kappa > 1$):

$$\lim_{r\to 0} r\,(w_k - w_0) \geq \lim_{r\to 0}\left(\Delta w_0 + \frac{rw_0 - (\pi_\infty - 2c\eta)}{a-1}\right)\frac{ra^k}{a-1} \geq \lim_{r\to 0}\left(\frac{2c\eta - \pi_\infty}{a-1}\right)\frac{ra^k}{a-1}.$$

Since $r\,(w_k - w_0)$ is bounded, it must be that ra^k is bounded; therefore $rk \sim 0$. QED.

Lemma A.2. $rv_{-k} \sim r\Delta v_{-k} \sim rv_{-n} \sim \Delta v_{-n} \sim 0$.

Proof. First, note that follower not investing in state $k+1$ implies $c \geq \Delta v_{-(k+1)}$. We apply Proposition A.1 to find an upper bound for $(v_{-n} - v_{-k})$ as a function of rv_{-k} and $\Delta v_{-(k+1)}$: $v_{-n} - v_{-k} \leq \lim_{r\to 0}\left(-\Delta v_{-(k+1)}\frac{\eta}{\eta-\kappa} + (n-k)\frac{rv_{-k}}{\eta-\kappa}\right)$, which implies $r(v_{-n} - v_{-k}) \sim 0$. Let $m \equiv \text{floor}(\frac{n+k}{2})$. That the follower does not invest in state m implies $c \geq \Delta v_{-m}$. Proposition A.1. provides a lower bound for $v_{-(n+1)} - v_{-n}$ as a function of rv_{-m} and Δv_{-m-1}: $\lim_{r\to 0}(v_{-(n+1)} - v_{-n}) \geq \lim_{r\to 0} -\Delta v_{-(m+1)}(\kappa/\eta)^{n-m} + \frac{rv_{-m}-\pi_{-m}}{\eta-\kappa} = \lim_{r\to 0}\frac{rv_{-m}}{\eta-\kappa}$, where the equality follows from $\lim_{r\to 0}(\kappa/\eta)^{n-m} = 0$ and $\lim_{r\to 0}\pi_{-m} \to 0$. Since the LHS is non-positive, it must be the case that $\lim_{r\to 0}\Delta v_{-n} = \lim_{r\to 0} rv_{-m} = 0$. But since $rv_{-n} \leq rv_{-m}$, it must be that $rv_{-n} \sim 0$, which, together with $rv_{-n} \sim rv_{-k}$, further implies $rv_{-k} \sim 0$. That $r\Delta v_{-k} \sim 0$ follows directly from the HJB equation for state k. QED.

We now prove $\lim_{r\to 0} k = \infty$**.** We show k bounded $\implies rw_k \sim r\Delta w_k \sim 0$, and we look for a contradiction. First, we use the fact that $0 \leq \pi_{-s}$ for all $0 \leq s \leq k$ and apply Proposition A.1 (simplification 1a, substituting $u_s \equiv v_{-k+1}$, $u_{s+t} = v_0$, $t = k+1$, $\Delta u_s = \Delta v_{-k}$, $a = \frac{\eta}{\eta+\kappa}$, $b = \frac{r}{\eta+\kappa}$, $\delta = \frac{rv_{-(k+1)}-(-c\eta)}{\eta+\kappa}$) to find an asymptotic upper bound for rv_0:

$$\lim_{r\to 0} rv_0 = \lim_{r\to 0} r\left(v_0 - v_{-(k+1)}\right) \leq \lim_{r\to 0}\frac{r}{1-\kappa/\eta}\left(\Delta v_{-(k+1)} + k\frac{rv_{-(k+1)}+c\eta}{\eta}\right)$$

By Lemma A.1(4) and Lemma A.2, the RHS converges to 0, implying $rv_0 \sim rw_0 \sim 0$. Further, using the HJB equation for state 0, we find that $\Delta w_0 \equiv w_1 - w_0 = \frac{rw_0+2c\eta-2\pi_0}{2\eta} \sim c - \pi_0/\eta$.

Lower and upper bounds for rw_k and $r\Delta w_k$ can be found, as functions of Δw_0 and rw_0, using Proposition A.1 (simplification 2(a), substituting $u_s \equiv w_0$, $u_{s+t} = w_k$, $t = k$, $\Delta u_s = \Delta w_0$, $a = \frac{\eta+\kappa}{\eta}$, $b = \frac{r}{\eta}$, and $\delta = \frac{rw_0-(-2c\eta)}{\eta}$ for the upper bound, $\delta = \frac{rw_0-(\pi_\infty-2c\eta)}{\eta}$ for the lower bound):

$$\lim_{r\to 0}\left(\Delta w_0 + \frac{rw_0+2c\eta-\pi_\infty}{\kappa}\right)\frac{\eta}{\kappa}r\left(\frac{\eta+\kappa}{\eta}\right)^k \leq \lim_{r\to 0}(rw_k - rw_0) \leq \lim_{r\to 0}\left(\Delta w_0 + \frac{rw_0+2c\eta}{\kappa}\right)\frac{\eta}{\kappa}r\left(\frac{\eta+\kappa}{\eta}\right)^k \tag{A.17}$$

$$\lim_{r\to 0}\left(\Delta w_0 + \frac{rw_0+2c\eta-\pi_\infty}{\kappa}\right)r\left(\frac{\eta+\kappa}{\eta}\right)^{k-1} \leq \lim_{r\to 0}(r\Delta w_k) \leq \lim_{r\to 0}\left(\Delta w_0 + \frac{rw_0+2c\eta}{\kappa}\right)r\left(\frac{\eta+\kappa}{\eta}\right)^{k-1}. \tag{A.18}$$

If k is bounded, these inequalities imply $rw_k \sim r\Delta w_k \sim 0$.

Now suppose $rw_k \sim r\Delta w_k \sim 0$ and we look for a contradiction. Let $\hat{k} \equiv \max\{k, N\}$ where N is the smallest integer such that $\pi_N - \pi_0 > c\kappa$. That $|N-k|$ is finite and $rw_k \sim r\Delta w_k \sim 0$ jointly imply $rw_N \sim r\Delta w_N \sim 0$. Note that $\pi_{\hat{k}}$ is a lower bound for π_s for all $n \geq s \geq \hat{k}$; we apply Proposition A.1 (simplification 1, substituting $u_s \equiv w_{\hat{k}}$, $u_{s+t} = w_{n+1}$, $t = n+1-\hat{k}$, $\Delta u_s = \Delta w_{\hat{k}}$, $a = \frac{\kappa}{\eta}$, $b = \frac{r}{\eta}$, $\delta = \frac{rw_{\hat{k}}-(\pi_{\hat{k}}-c\eta)}{\eta}$) and obtain $\frac{rw_{\hat{k}}-(\pi_{\hat{k}}-c\eta)}{\eta-\kappa}$ as an asymptotic upper bound for $w_{n+1} - w_n$. Lemma A.1 part 2 further implies that

$$\lim_{r\to 0}\frac{rw_{\hat{k}} - (\pi_{\hat{k}} - c\eta)}{\eta-\kappa} \geq c \iff \lim_{r\to 0} rw_{\hat{k}} \geq \pi_{\hat{k}} - c\kappa > 0. \tag{A.19}$$

This contradicts the presumption that $rw_{\hat{k}} \sim 0$. QED.

Note that (A.17), (A.18), and the contradiction above jointly imply $\lim_{r\to 0} rw_k > 0$, $\lim_{r\to 0} r\Delta w_k > 0$, and that $r\left(\frac{\eta+\kappa}{\eta}\right)^k$ converges to a positive constant, summarized as a Lemma.

Lemma A.3. $\lim_{r\to 0} r\Delta w_k > 0$, and $r\left(\frac{\eta+\kappa}{\eta}\right)^k$ converges to a positive constant as $r \to 0$.

Proof of Theorem 5.5. We show $\lim_{r\to 0} (\kappa/\eta)^{n-k} (1+\kappa/\eta)^k = 0$, which, based on Lemma 4.5, is a sufficient condition for $\mu^M \to 1$, $\mu^C \to 0$, and $g \to \kappa \cdot \ln \lambda$.

We first find a lower bound for Δw_k by applying simplification 2 of Proposition A.1 (substituting $u_s \equiv w_0$, $u_{s+t} = w_k$, $t = k$, $\Delta u_s = \Delta w_0$, $a = \frac{\eta+\kappa}{\eta}$, $b = \frac{r}{\eta}$, $\delta = \frac{rw_0 - (\pi_\infty - 2c\eta)}{\eta}$):

$$\lim_{r\to 0} r\Delta w_k \geq \lim_{r\to 0} \left(\Delta w_0 + \frac{rw_0 - (\pi_\infty - 2c\eta)}{\kappa}\right) r \left(\frac{\eta+\kappa}{\eta}\right)^k. \tag{A.20}$$

Simplification 1 of Proposition A.1 provides asymptotic bounds for Δw_n (substituting $u_s = w_k$, $u_{s+t} = w_n$, $t = n-k$, $\Delta u_s = \Delta w_k$, $a = \frac{\kappa}{\eta}$, $b = \frac{r}{\eta}$; the upper bound is obtained using $\delta = \frac{rw_k - (\pi_k - c\eta)}{\eta}$ and the lower bound is obtained using $\delta = \frac{rw_k - (\pi_\infty - c\eta)}{\eta}$):

$$\lim_{r\to 0} \left[\Delta w_k \left((\kappa/\eta)^{n-k} + \frac{r\eta}{(\eta-\kappa)^2}\right) + \frac{rw_k + c\eta - \pi_k}{\eta - \kappa}\right] \geq \lim_{r\to 0} \Delta w_n$$

$$\lim_{r\to 0} \Delta w_n \geq \lim_{r\to 0} \left[\Delta w_k \left((\kappa/\eta)^{n-k} + \frac{r\eta}{(\eta-\kappa)^2}\right) + \frac{rw_k + c\eta - \pi_\infty}{\eta - \kappa}\right].$$

Since $\lim_{r\to 0} \pi_k = \pi_\infty$, the lower and upper bounds coincide asymptotically. Furthermore, Lemma A.1 shows $\Delta w_n \sim c$; hence,

$$c \sim \Delta w_k \left((\kappa/\eta)^{n-k} + \frac{r\eta}{(\eta-\kappa)^2}\right) + \frac{rw_k + c\eta - \pi_\infty}{\eta - \kappa}. \tag{A.21}$$

Next, we apply simplification 1(b) of Proposition A.1 to obtain (substituting $u_s \equiv w_k$, $u_{s+t} = w_n$, $t = n-k$, $\Delta u_s = \Delta w_k$, $a = \frac{\kappa}{\eta}$, $b = \frac{r}{\eta}$; the simplification applies because $\lim_{r\to 0} r\Delta w_k > 0$, as stated in Lemma A.3): $r(w_n - w_k) \sim \frac{r\Delta w_k}{(\eta-\kappa)/\eta}$. Part 1 of Lemma A.1 further implies

$$\pi_\infty - c\kappa - rw_k \sim \frac{r\Delta w_k}{(\eta-\kappa)/\eta}. \tag{A.22}$$

Substituting the asymptotic equivalence (A.22) into (A.21), we obtain

$$c \sim c + \Delta w_k \left((\kappa/\eta)^{n-k} + \frac{r\eta}{(\eta-\kappa)^2}\right) - \frac{r\eta\Delta w_k}{(\eta-\kappa)^2}$$

$$\implies 0 \sim \Delta w_k (\kappa/\eta)^{n-k}.$$

Further substitute into inequality (A.20),

$$0 \geq \lim_{r\to 0} \left(\Delta w_0 + \frac{rw_0 - (\pi_\infty - 2c\eta)}{\kappa}\right) \left(\frac{\eta+\kappa}{\eta}\right)^k (\kappa/\eta)^{n-k}$$

Given $\Delta w_0 \geq 0$, $rw_0 \geq 0$, and $2c\eta - \pi_\infty > 0$, the inequality holds if and only if $\lim_{r\to 0}\left(\frac{\eta+\kappa}{\eta}\right)^k (\kappa/\eta)^{n-k} = 0$, as desired. All other claims in Theorem 5.5 follows directly. QED.

Finally, the next result characterizes the relative rate of divergence between $(n-k)$ and k, as well as the rate of convergence of μ^M.

Lemma A.4. 1) $\lim_{r\to 0}\frac{n-k}{k} = \frac{2\ln(1+\kappa/\eta)}{\ln \eta/\kappa}$; 2) $\lim_{r\to 0}\frac{1-\mu^M}{r}$ goes to a positive constant.

Proof of Lemma A.4. We first prove $\frac{n+k}{k} \sim \frac{2\ln(1+\alpha)}{-\ln \alpha}$. Note Lemmas A.1 and A.2 jointly imply $\frac{rw_{n+1}-(\pi_\infty - c\eta)}{\eta-\kappa} \sim c \sim \Delta w_n$. We apply Proposition A.1 simplification 2(b) to find $\lim_{r\to 0} rw_k$. We substitute $u_s = w_{n+1}$, $u_{s+t} = w_k$, $\Delta u_s = w_n - w_{n+1} = -\Delta w_n$, $a = \frac{\eta}{\kappa}$, $b = \frac{r}{\kappa}$; the upper bound is obtained using $\delta = \frac{rw_{n+1}-(\pi_k - c\eta)}{\kappa}$ and the lower bound is obtained using $\delta = \frac{rw_{n+1}-(\pi_\infty - c\eta)}{\kappa}$, and that the lower and upper bounds coincide as $r \to 0$. Simplification 2(b) applies because $\Delta u_s + \frac{\delta}{a-1} \sim -c + \frac{rw_{n+1}-(\pi_\infty - c\eta)}{\kappa(\eta/\kappa - 1)} \sim 0$. Proposition A.1 implies

$$w_k - w_{n+1} \sim -\frac{r}{\kappa(\eta/\kappa - 1)^4}\frac{c(\eta-\kappa)}{\kappa}(\eta/\kappa)^{n+1-k}$$

$$\implies r(w_{n+1} - w_k) \sim \frac{c(\eta-\kappa)}{\kappa^2(\eta/\kappa - 1)^4} r^2 (\eta/\kappa)^{n+1-k}$$

$$\text{substitute into } (A.22) \implies r\Delta w_k \sim \varphi_1 \cdot r^2 (\eta/\kappa)^{n-k} \quad \text{for some constant } \varphi_1 > 0.$$

We denote $a = \Phi(f(r))$ if $a/f(r)$ converges to a positive constant as $r \to 0$. By Lemma A.3, $\lim_{r\to 0} r\Delta w_k > 0$, hence $(\kappa/\eta)^{n-k} = \Phi(r^2)$. Lemma A.3 also states that $(1+\kappa/\eta)^{-k} = \Phi(r)$; hence $(\eta/\kappa)^{n-k} \sim \varphi_2 (1+\kappa/\eta)^{2k}$ for some constant $\varphi_2 > 0$, implying

$$(n-k)\ln(\eta/\kappa) \sim \ln\varphi_2 + 2k\ln\left(\frac{\eta+\kappa}{\eta}\right)$$

$$\implies \frac{n-k}{2k} \sim \frac{2\ln(1+\kappa/\eta)}{\ln \eta/\kappa}, \quad \text{as desired.}$$

We now prove $1 - \mu^M = \Phi(r)$. By Lemma 4.5 and denoting $\alpha \equiv \kappa/\eta$,

$$1 - \mu^M = \frac{\alpha^{n-k}\left((1+\alpha)^k - 1\right) + \alpha^{n-k+1}(1+\alpha)^k/2}{\frac{1-\alpha^{n-k+1}}{1-\alpha} + \alpha^{n-k}\left((1+\alpha)^k - 1\right) + \alpha^{n-k+1}(1+\alpha)^k/2}.$$

Hence $(1-\mu^M) \sim (\kappa/\eta)^{n-k}(1+\kappa/\eta)^k$. But we have established above that $(\kappa/\eta)^{n-k} = \Phi(r^2)$ and $(1+\kappa/\eta)^{-k} = \Phi(r)$; jointly, these relationships imply $1-\mu^M = \Phi(r)$, as desired.

APPENDIX B: EXTENSIONS

B.1. *General Equilibrium Extension*

In this Appendix we embed the partial equilibrium model in Sections 4 and 5 into a general equilibrium framework. We focus on a steady-state equilibrium, i.e., a balanced growth path, with aggregate productivity and consumption both growing at a constant rate g. We start with a discrete time economy and take the limit as the time between periods shrinks to zero to match the continuous-time setting as in the paper. There is unit measure of households, each with intertemporal preferences:

$$\max_{\{y_1(t;\nu),y_2(t;\nu),\ell(t)\}} \mathbb{E}\sum_{t=0}^{\infty} e^{-\rho t}\left(\ln c_t - \ell_t\right) \tag{A.23}$$

$$\text{s.t. } c_t = \exp\left(\int_0^1 \ln\left[y_1(t;\nu)^{\frac{\sigma-1}{\sigma}} + y_2(t;\nu)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} d\nu\right),$$

$$\int_0^1 p_1(t;\nu)\,y_1(t;\nu) + p_2(t;\nu)\,y_2(t;\nu)\,d\nu + \frac{b_{t+1}}{1+\hat{r}_t} = \left[\zeta_t w_t \ell_t + d_t + T_t\right] + b_t$$

where $e^{-\rho}$ is the discount rate, d_t is dividends income, b_t is the holding of a risk-free bond. We assume households hold the same market portfolio of all firms and therefore receive identical dividend payments that is equal to the total flow payoff that firms receive in a period. We normalize the wage rate $w_t \equiv 1$ for all t, and specify that production and the investment cost are both paid in labor.

To generate variations in the interest rate, we follow Benigno and Fornaro (2018) and introduce uninsurable, idiosyncratic unemployment risk, captured by ζ_t, which is an indicator variable that takes value 1 if the household is employed, and zero if the household is unemployed. Each household faces in every period a constant probability q of being unemployed, and the employment status is revealed at the start of the period. T_t is a lump-sum transfer for the unemployed households and a tax for employed households. T_t is set such that the income of an unemployed household is equal to a fraction $\delta < 1$ of the income of an employed household.[1] We further assume unemployed households cannot borrow ($b_{t+1} \geq 0$) and that trade in firms' shares is not possible, so that every household receives the same dividends.

The labor market clearing condition is

$$\begin{aligned} q\ell(t) = & \int_0^1 \left[y_1(t;\nu)\,\lambda^{-z_1(t;\nu)} + y_2(t;\nu)\,\lambda^{-z_2(t;\nu)}\right] d\nu \\ & + \left(\sum_{s=1}^{\infty} \mu_s(t)\left(c(\eta_s) + c(\eta_{-s})\right) + 2\mu_0(t)\,c(\eta_0)\right). \end{aligned}$$

The consumption aggregator $c(t)$ in (A.23) features CES across varieties within each market and Cobb-Douglas across markets. Given our normalization $w_t = 1$, the employed households' intratemporal problem implies total expenditure on all consumption goods is constant along the balanced growth path, thereby inducing instantaneous demand functions that coincide with the preferences in (4) of Section 4 subject to a normalization in the level of expenditure. As in Benigno and Fornaro (2018), the Euler equation of the employed implies an aggregate relationship between consumption growth and the interest rate:

$$\frac{c_{t+1}}{c_t} = e^{-\rho t}\left(1+\hat{r}_t\right)\left(1 - q + q/\delta\right)$$

The term $(1 - q + q/\delta)$ captures the precautionary saving incentive for employed households due to idiosyncratic unemployment risk. Taking logs and then take the limit as the time between

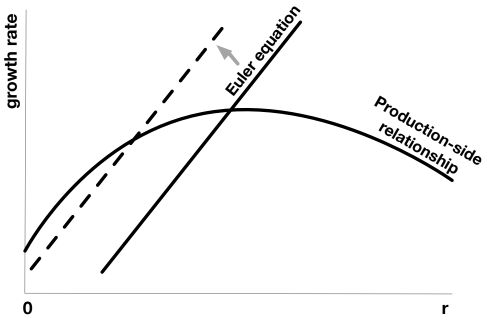
[1] That is, an unemployed receives a transfer $T_t = \frac{b(w_t\ell_t+d_t)-d_t}{1+bq/(1-q)}$ while an employed pays a tax $T_t = -\frac{q}{1-q}\frac{b(w_t\ell_t+d_t)-d_t}{1+bq/(1-q)}$.

periods shrinks to zero, we obtain the continuous-time analogue:

$$g(t) \equiv \frac{\mathrm{d}\ln c(t)}{\mathrm{d}t} = \hat{r}(t) - \xi \tag{A.24}$$

where $\xi \equiv \rho + q - q/\delta < \rho$. The interest rate and the growth rate along the balanced growth path is jointly pinned down by the demand-side relationship (A.24) and a production-side relationship derived in the partial equilibrium model of Sections 4 and 5. A reduction in the interest rate driven by demand-side forces, such as increasing patience or heightened levels of uninsurable risk, can be modeled as a decline in ξ, as depicted in Figure B.1.

FIGURE B.1.— Growth and the interest rate in general equilibrium



On a balanced growth path, the consumption price index $P(t)$ takes the same form as defined in Section 4.1, and it declines at a constant rate g relative to the numeraire; hence, the value function of a firm currently in state s is

$$\begin{aligned} v_s(t) &= \mathbb{E}\left[\int_0^\infty e^{-\hat{r}\tau}\left\{\frac{\pi(t+\tau) - c(t+\tau)}{P(t+\tau)/P(t)}\right\} d\tau \Big| s\right] \\ &= \mathbb{E}\left[\int_0^\infty e^{-(\hat{r}-g)\tau}\left\{\pi(t+\tau) - c(t+\tau)\right\} d\tau \Big| s\right]. \end{aligned}$$

The general equilibrium version of the main result of Theorem 5.5 states that, as ξ declines towards zero, $\hat{r} - g \to 0$, and aggregate productivity growth rate g must decline and converge to $\kappa \cdot \ln \lambda$.[2]

B.2. *Asset-Pricing Implications*

Our key mechanism implies that, starting from a steady-state with a low interest rate, a further decline in r raises the expected future cash flows of current market leaders by causing their leadership to become more persistent. Hence, a decline in r raises the firm value of market leaders relative to followers, and the asymmetric valuation response is larger at lower levels

[2]The reason we introduce unemployment risk á la Benigno and Fornaro (2018) is as follows. Absent unemployment risk ($q = 0$), the Euler equation takes the standard form given log-utility over c_t: $g = \hat{r} - \rho$. In that case, a decline in ρ can still generate a decline in the interest rate, but there is a lower bound on ρ below which the consumer optimization problem is no longer well-defined (i.e., when consumption growth is higher than the rate of discounting), and $\hat{r} - g$ cannot fall to zero. Unemployment risk creates a channel for the interest rate to fall close to zero.

of the interest rate. In this appendix, we formalize this intuition into a prediction testable using asset prices. Let us start with a steady-state economy with interest rate r and consider an unexpected and permanent decline in the interest rate $-\mathrm{d}r$. Lower discounting of future cash flows raises market values of all firms; moreover, investment decisions respond endogenously, further affecting firm valuations. We focus on the immediate, on-impact effect of the shock on the *relative* firm value between leaders and followers. Let v_s and $\hat{v}_s$ respectively denote the pre- and post-shock value function in state s. Define $\frac{\hat{V}^L}{V^L} \equiv \frac{\sum_{s=0}^{\infty} \mu_s \hat{v}_s}{\sum_{s=0}^{\infty} \mu_s v_s}$. The numerator evaluates leaders' market value in the new equilibrium using the productivity gap distribution from the pre-shock steady-state; therefore, $\mathrm{d}\ln V^L \equiv \frac{\hat{V}^L}{V^L} - 1$ captures the *on-impact* effect of the interest rate shock $-\mathrm{d}r$ on the total value of market leaders, before the economy starts transitioning to the new, post-shock steady-state. We define $\frac{\hat{V}^F}{V^F}$ and $\mathrm{d}\ln V^F$ analogously for followers.

PROPOSITION 1: *Consider a steady-state with interest rate r. To first-order around $r = 0$, a permanent change in the interest rate has the following on-impact, proportional effect on the valuation of leaders and followers:*

$$-\frac{\mathrm{d}\ln V^L}{\mathrm{d}r} = \frac{1}{r} \quad \textit{and} \quad -\frac{\mathrm{d}\ln V^F}{\mathrm{d}r} = \frac{1}{-r\ln r}.$$

The proposition states that, starting from a steady-state with low r, a small decline in the interest rate $-\mathrm{d}r$ immediately raises leaders' market value by a proportion of $1/r$ and raises followers' value by a proportion of $1/(r\ln r)$. The relative valuation response between leaders and followers, $-\frac{\mathrm{d}\ln\left(V^L/V^F\right)}{\mathrm{d}r} = \frac{1}{r}\left(1 + \frac{1}{\ln r}\right)$, increases and diverges to infinity as $r \to 0$. This is an empirically-testable prediction. Starting from a low-r steady-state and following an unexpected further decline in the interest rate, market leaders at the time of the shock should experience immediate valuation gains relative to market followers. The asymmetric valuation effect should be more pronounced when the pre-shock interest rate is lower.

Importantly, low r affects relative firm valuations not only through changing the discount rate but also through changes in future cash flows that favor the current leaders. Holding cash-flows constant, followers in the model expect more distant payoff streams, and their firm value should therefore be more sensitivity to changes in the interest rate. However, because investments respond endogenously to interest rates, cash flows are expected to change. Leaders tend to raise investments more than followers do. The endogenous investment response increases leader's duration and the persistence of market power. Changes in future cash flows are key in explaining why leaders may have longer duration than followers. These predictions would not emerge naturally from other models, and they form a powerful test of our model's dynamics.

The proposition shows that, if the interest rate declines from an already low level, the endogenous investment response dominates the mechanical duration effect, and therefore leader value unambiguously increases more than follower value. To understand why the asymmetry is stronger when r is lower, note that the valuation responses of leaders and followers depend on the state variable in the respective industries. When the leader-follower gap is small—the state is competitive and close to neck-and-neck—a lower interest rate may actually reduces the leader's value relative to the follower's, as maintaining leadership becomes more difficult due to followers' investment response. On the other hand, when the leader-follower gap is sufficiently large, the follower invests little as it is discouraged, and a lower interest rate boosts the relative value of leaders even further because the far-ahead leaders now expects even more persistent profits due to the asymmetric investment response. Proposition 2 aggregates these state-by-state valuation effects to the entire economy. If the initial interest rate is high, the steady-state features a significant mass of markets in which the follower stays competitive, and the average leader in the economy experiences a valuation loss relative to the average follower if the interest rate declines. Conversely, starting from a low-r steady-state, the distribution of the state variable concentrates in regions in which the follower is no longer competitive, and, therefore, the average leader experiences valuation gains relative to the average follower in the economy when the interest rate declines. The lower is the initial interest rate, the stronger is this asymmetry. To prove the Proposition, we first establish a Lemma.

Lemma A.5. $\Delta v_{-k} \sim c$, $v_{-k} \sim \frac{c}{1-\kappa/\eta}$, $v_{-(n+1)} \sim 0$. Proof. Note that $v_{-(k-1)} - v_{-k} \geq c$, $v_{-(k-2)} - v_{-(k-1)} \geq c$, and $c \geq v_{-k} - v_{-(k+1)}$. Substitute these inequalities into the HJB equations for followers in state $k-1$ and k, we get

$$\left(v_{-(k-1)} - v_{-k}\right) \leq \frac{\pi_{-(k-1)} - \pi_{-k}}{2\eta + \kappa + r} + \frac{2\eta + \kappa}{2\eta + \kappa + r} c,$$

which implies $\lim_{r \to 0} \left(v_{-(k-1)} - v_{-k}\right) \leq c$. Coupled with the fact that $v_{-(k-1)} - v_{-k} \geq c$, this establishes that $v_{-(k-1)} - v_{-k} \sim c$. That $v_{-k} - v_{-(n+1)} \sim \frac{c}{1-\kappa/\eta}$ can be obtained by applying simplification 1a) of Proposition A1. It remains to show $v_{-(n+1)} \sim 0$. Note that we can write $v_{-(n+1)}$ as a weighted average of the flow payoffs in states $k+1$ through $n+1$ and the value function in state $-k$:

$$v_{-(n+1)} = \textstyle\sum_{s=k+1}^{n+1} \epsilon_s \pi_{-s} + \epsilon_k v_{-k}, \quad \text{where } \textstyle\sum_{s=k}^{n} \epsilon_k = 1.$$

The flow payoffs π_{-k} approach zero as $r \to 0$; hence, $v_{-(n+1)} \sim \epsilon_k v_{-k}$. The term ϵ_k can be found by solving the recursive relationship

$$\begin{aligned}
v_{-(n+1)} &= \frac{\kappa}{r+\kappa} v_{-n} \\
v_{-n} &= \frac{\kappa}{r+\kappa+\eta} v_{-(n-1)} + \frac{\eta}{r+\kappa+\eta} v_{-(n+1)} \\
&\vdots \\
v_{-(k+1)} &= \frac{\kappa}{r+\kappa+\eta} v_{-k} + \frac{\eta}{r+\kappa+\eta} v_{-(k+2)}.
\end{aligned}$$

It is easy to see that $\epsilon_k < (\kappa/\eta)^{n-k}$; hence, as $r \to 0$, $\frac{v_{-(n+1)}}{v_{-k}} \to 0$. This implies that $v_{-(n+1)} \sim 0$ and $v_{-k} \sim \frac{c}{1-\kappa/\eta}$, as desired. QED.

Proof of Proposition 2. Let (k, n) be the equilibrium investment decisions under interest rate r and (k_2, n_2) be the investments under $r - \mathrm{d}r$. Recall $\alpha \equiv \kappa/\eta$. We now show $\mathrm{d}\ln V^F = \frac{k_2-k}{k} + O(r)$. The total market value of followers is

$$\textstyle\sum_{s=1}^{k} \mu_s v_{-s} + \sum_{s=k+1}^{n+1} \mu_s v_{-s} = 2\mu_0 \left(\sum_{s=1}^{k} a^s v_{-s}\right) + \mu_{k+1} \left(\sum_{s=0}^{n-k} b^s v_{-(k+1+s)}\right)$$

where $a \equiv \frac{\eta}{\eta+\kappa}$ and $b \equiv \eta/\kappa$. We analyze the two terms on the RHS separately. First, we show the total value of followers in the competitive region scales with k asymptotically, i.e., $\sum_{s=1}^{k} \mu_s v_{-s} \sim Ck$ for some constant C. For any $m < k$, we can write $\sum_{s=1}^{k} \mu_s v_{-s}$ as

$$\textstyle\sum_{s=1}^{k} \mu_s v_{-s} = 2\mu_0 \left(\sum_{s=1}^{m-1} a^s v_{-s} + \sum_{s=m}^{k} a^s v_{-s}\right)$$

Lemma A.5 shows $\Delta v_{-k} \sim c$ and $v_{-k} \sim \frac{c}{1-\kappa/\eta}$. For any $s' \geq m$, we apply Proposition A.1 to generate asymptotic upper- and lower-bounds for $v_{-s'}$ and $\Delta v_{-s'}$. Specifically, let $\overline{v_{-s'}} \equiv \frac{c}{1-a}\left(1-a^{k-s'}\right) + \frac{ca}{1-a}\left((k-s') - \frac{a-a^{k-s'}}{1-a}\right)$, $\underline{v_{-s'}} \equiv \frac{c}{1-a}\left(1-a^{k-s'}\right) + \frac{ca-\frac{\pi_m}{\eta+\kappa}}{1-a}\left((k-s') - \frac{a-a^{k-s'}}{1-a}\right)$, $\overline{\Delta v_{-s'}} \equiv ca^{k-s'} + ca\frac{\left(1-a^{k-s'-1}\right)}{1-a}$, and $\underline{\Delta v_{-s'}} \equiv ca^{k-s'} + \left(ca - \frac{\pi_m}{\eta+\kappa}\right)\frac{\left(1-a^{k-s'-1}\right)}{1-a}$. Then

$$\lim_{r\to 0}\left(\overline{v_{-s'}} - v_{-s'}\right) \qquad \lim_{r\to 0}\left(v_{-s'} - \underline{v_{-s'}}\right),$$
$$\lim_{r\to 0}\left(\overline{\Delta v_{-s'}} - \Delta v_{-s'}\right) \geq 0 \geq \lim_{r\to 0}\left(\underline{\Delta v_{-s'}} - \Delta v_{-s'}\right).$$

Analogously for all $s < m$, we apply Proposition A.1 to find bounds for v_{-s} using $\overline{v_{-m}}$, $\underline{v_{-m}}$, $\overline{\Delta v_{-m}}$, and $\underline{\Delta v_{-m}}$. Specifically, let $\overline{v_{-s}} \equiv \overline{v_{-m}} + \overline{\Delta v_{-m}}\frac{1-a^{m-s}}{1-a} + \frac{ca-\frac{\pi_m}{\eta+\kappa}}{1-a}\left(m-s-\frac{a-a^{m-s}}{1-a}\right)$ and $\underline{v_{-s}} \equiv \underline{v_{-m}} + \underline{\Delta v_{-m}}\frac{1-a^{m-s}}{1-a} + \frac{ca-\frac{\pi_s}{\eta+\kappa}}{1-a}\left(m-s-\frac{a-a^{m-s}}{1-a}\right)$, then $\lim_{r\to 0}\left(\overline{v_{-s}} - v_{-s}\right) \geq 0 \geq \lim_{r\to 0}\left(\underline{v_{-s}} - v_{-s}\right)$. Using these bounds for v_{-s}, we can now find upper and lower-bounds for $\sum_{s=1}^{k}\mu_s v_{-s}$:

$$0 \leq \lim_{r\to 0} 2\mu_0\left(\sum_{s=1}^{m-1} a^s\overline{v_{-s}} + \sum_{s'=m}^{k} a^s\overline{v_{-s'}}\right) - \sum_{s=1}^{k}\mu_s v_{-s}$$
$$0 \geq \lim_{r\to 0} 2\mu_0\left(\sum_{s=1}^{m-1} a^s\underline{v_{-s}} + \sum_{s'=m}^{k} a^s\underline{v_{-s'}}\right) - \sum_{s=1}^{k}\mu_s v_{-s}$$

These bounds simplifies to

$$0 \leq \lim_{r\to 0}\left(2\mu_0 c\left(\frac{a}{1-a}\right)^2 k - \sum_{s=1}^{k}\mu_s v_{-s}\right)$$
$$0 \geq \lim_{r\to 0}\left(2\mu_0\left(c-\pi_m/\eta\right)\left(\frac{a}{1-a}\right)^2 k - \sum_{s=1}^{k}\mu_s v_{-s}\right).$$

Since m is arbitrarily chosen, π_m can be made arbitrarily close to zero; hence we conclude $\sum_{s=1}^{k}\mu_s v_{-s} \sim 2\mu_0 c\left(\frac{a}{1-a}\right)^2 k$. We now compute the market value of followers in the monopolistic region. Using Proposition A1, we derive

$$v_{-(k+s)} \sim v_{-k} - \frac{c}{1-\kappa/\eta}\left(1-(\kappa/\eta)^s\right) \sim \frac{c}{1-\kappa/\eta}(\kappa/\eta)^s,$$

$$\Longrightarrow \sum_{s=k+1}^{n+1}\mu_s v_{-s} = \mu_{k+1}\sum_{s=0}^{n-k}(\eta/k)^s v_{-(k+1+s)} \sim \mu_{k+1}\frac{\alpha c}{1-\alpha}(n-k)$$

The total market value of followers is thus

$$V^F \equiv \sum_{s=1}^{k}\mu_s v_{-s} + \sum_{s=k+1}^{n+1}\mu_s v_{-s} \sim 2\mu_0 c\left(\frac{a}{1-a}\right)^2 k.$$

Now consider the new equilibrium characterized (k_2, n_2) under interest rate $r - \mathrm{d}r$. Let value functions be denoted by $\hat{v}_s$ under the new equilibrium. The market value of followers, evaluated using the steady-state under r, is $\hat{V}^F \equiv \sum_{s=1}^{k} \mu_s \hat{v}_{-s} + \sum_{s=k+1}^{n+1} \mu_s \hat{v}_{-s}$. Following the same derivation as before, we can show $\hat{V}^F \sim 2\mu_0 c \left(\frac{a}{1-a}\right)^2 k_2$, thus $\mathrm{d}\ln V^F \equiv \frac{\hat{V}^F}{V^F} - 1 = \frac{k_2 - k}{k} + O\left(r\right)$. That $\mathrm{d}\ln V^F = \frac{\log(r - \mathrm{d}r)}{\log r} + O\left(r\right) \iff -\frac{\mathrm{d}\ln V^F}{\mathrm{d}r} = \frac{1}{r\ln r}$ follows from the convergence of $r\left(\frac{\eta+\kappa}{\eta}\right)^k$ to a positive constant (Lemma A.3).

The on-impact, proportional change in the total market value of leaders can be derived analogously, as Proposition A.1 enables us to derive an asymptotic analytic approximation for the value functions. We omit the derivations here and instead provide a simpler intuition for the result. As interest rate converges to zero, the total market value of leaders becomes inversely proportional to the interest rate (rV^L converges to a positive constant). Hence, following a small decline in interest rate, the value of leaders changes proportionally with the interest rate, i.e. $-\frac{\mathrm{d}\ln V^L}{\mathrm{d}r} = \frac{1}{r}$.

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