A Dynamic Model of Input-Output Networks*

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Abstract

We develop a dynamic model of input-output networks that incorporates adjustment costs of changing inputs. Our closed-form solution for the dynamics of the economy shows that temporary shocks to upstream sectors, whose output travels through long supply chains, have disproportionately significant welfare impact compared to affected sectors' Domar weights. We conduct a spectral analysis of the U.S. production network and reveal that the welfare impact of temporary sectoral shocks can be represented by a low-dimensional, 4-factor structure.

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1 Introduction

We develop a dynamic model of input-output networks that generalizes the static setting of Acemoglu et al. (2012). The key feature of our environment is the adjustment costs of changing inputs, which result in a gradual recovery in the input-output linkages following a productivity shock. This gradual recovery introduces rich dynamics in the transmission of shocks across sectors, where even temporary shocks may have long-lasting effects. We derive a closed-form expression for the dynamic path of shock propagation through input-output linkages, enabling a detailed analytical characterization of the factors that influence output trajectory and welfare impact. We show that, due to dynamic adjustments, sectors critical for temporary shocks differ from those important for permanent shocks and for the static model, and that the exact structure of the input-output network is essential for examining both shock propagation and welfare impact. Specifically, the economy takes longer to recover from temporary shocks to upstream sectors, as their output goes through lengthy supply chains before reaching the final consumer. Consequently, these shocks have disproportionately significant welfare impact compared to the affected sectors' Domar weights.

Our tractable formulation of the dynamic model allows us to derive explicit, closed-form solution for the time path of efficient input-output allocations.[1] We first characterize the transition path of sectoral output and aggregate consumption. To isolate the dynamics that arise from adjustment costs of the endogenous state variables, we examine an economy that have experienced temporary negative sectoral TFP shocks, disrupting input-output linkages. We derive the recovery dynamics of these linkages after TFP recovers. Due to adjustment costs, the use of intermediate goods can only recover gradually over time. The gradual recovery of inputs then translates into the gradual recovery of the output in sectors that use those inputs. As the transition takes time, the speed of which is determined by the magnitude of adjustment costs, the path of adjustment has non-trivial welfare implications

[1]The primary technical challenge that we address is the incorporation of input-output adjustment cost which results in slow-moving input-output quantities and the model therefore features an entire matrix of endogenous state variables, one for each input-output pair of sectors.

for a consumer who discounts the future.

The steady state of our economy features the same production technology as the static model. Hence, when shocks are permanent, the Domar weight—a measure of sectoral size in the static model of Acemoglu et al. (2012)—reflects the elasticity of the steady-state output to TFP shocks. By contrast, when shocks are temporary, the welfare impact of sectoral shocks during the recovery depends crucially on the particular details of the network structure. We show the Domar weight alone is insufficient for characterizing the welfare impact of temporary shocks in our dynamic model. Instead, the welfare impact is determined by the product of the Domar weight and a measure of upstreamness. A sector is more upstream if a larger fraction of its output passes through long supply chains before reaching the final consumer. Intuitively, shocks have greater welfare consequences if they generate both large and long-lasting impact. Shocks to high Domar weight sectors generate large short-run impact during the recovery while shocks to upstream sectors generate long-lasting impact due to adjustment costs. In other words, the Domar weight is the main determinant of the short-run impact, and the upstreamness is the main determinant of the persistent dynamic impact.

While the main part of the paper assumes exponential adjustment costs, we extend our model to a general constant-returns-to-scale formulation. Our closed-form characterization continues to hold in this general setting as a first-order approximation of the general economy around its steady-state. We further extend our baseline analysis to production networks featuring 1) heterogeneous adjustment costs across sectors, 2) gradual contraction as well as expansion of inputs, and 3) continuous recovery of productivity shocks.

We conduct an empirical analysis of the U.S. input-output network using our model. We compute the welfare impact of temporary shocks, and we show that our measure of sectoral importance given by the product of the Domar weight and upstreamness differs significantly from simply the Domar weights. While Domar weights emphasize large sectors such as hospitals, restaurants, and retail, our measure selects both large and upstream sectors such as motor vehicle parts manufacturing, basic chemical manufacturing, and sectors that provides services to commercial activities (agencies, brokerages, and insurance). We then show that, given the rich structure of the U.S. input-output network, sectoral shocks have significantly heterogeneous dynamic properties. The GDP recovers quickly from shocks to finance, oil and gas extraction, and petroleum and coal products manufacturing—three large sectors that are traditionally viewed as important because of their high Domar weights—with half-lives averaging to 4.6 months. By contrast, the average half-life is more than twice as long for GDP recovery from shocks to the manufacturing of communication equipments, motor vehicles, and motor vehicle parts—three heavy-manufacturing sectors that are relatively upstream.

We next characterize the main driving forces of the welfare impact of temporary shocks. Since we have shown the importance of the higher-round network effects, this leads us to analyze the spectral decomposition of the input-output matrix—its eigenvectors and eigenvalues. When the direct impact of TFP shocks coincides with an eigenvector, each round of the network effect of the shock is a proportionally decayed version of the initial impact, with the rate of decay governed by the eigenvalue. An important corollary of this logic shows a marked contrast with the Domar weights, and thus with the static economy. The dynamic adjustment costs significantly down-weight the importance of the direct and initial rounds of network effects along the recovery path, as these effects recover quickly and are not long-lasting. This implies that significantly fewer factors are needed to represent the welfare impact of temporary shocks in a dynamic economy. By contrast, almost all eigenvectors are needed to explain the variation in the Domar weights. This is because the Domar weights does not discount the direct and initial rounds of network effects, and even eigenvectors with small eigenvalues may have a sizable contribution in explaining TFP shocks in the static model.

We conduct a spectral analysis of the U.S. production network. We first show that 95 percent of the welfare effect of temporary sectoral shocks can be represented by the low-dimensional projection onto four eigenvectors (out of 171). That is, the U.S. input-output network has a low-dimensional (4-factor) structure. In contrast, for the Domar weight almost all of the 171 eigenvectors are important. We identify the groups of sectors that form the four key eigenvectors. The first eigenvector represents shocks to the heavy manufacturing sectors. The second eigenvector additionally represents sectors relating to agencies, brokerages, and insurance and sectors covering the manufacturing of consumer goods. The third eigenvector picks up sectors representing chemicals. The fourth eigenvector represents entertainment, including radio and television broadcasting. Summarizing, we find that the welfare impact of any negative temporary shocks can be represented by only four (out of 171) eigenvectors.

Literature

There is a modern revival of the literature on production networks (see, e.g., reviews in macroeconomics of Carvalho (2014), Carvalho and Tahbaz-Salehi (2019), and Grassi and Sauvagnat (2019)). Carvalho (2010), Gabaix (2011) and Acemoglu et al. (2012) show idiosyncratic sectoral productivity shocks may have aggregate impact. Jones (2011, 2013) develops a model of production networks with distortions. A number of papers develop important aspects of the macroeconomic implications of the input-output and production structure of the economy: for example, Grassi (2017), Baqaee (2018), Oberfield (2018), Liu (2019), Baqaee and Farhi (2019, 2020), Golub et al. (2020), vom Lehn and Winberry (Forthcoming). Our contribution to the literature is to develop a model of dynamic production networks. The welfare impact of productivity shocks in our setting differs from Acemoglu et al. (2012) due to adjustment costs and the slow-recovery of input-output linkages. This is distinct from other departures of the static setting studied in the literature such as non-linearities in the production technology (Baqaee and Farhi, 2019) and allocative inefficiency (Liu 2019, Baqaee and Farhi 2020). Liu and Ma (2021) adapt our framework and build an endogenous growth model with an innovation network featuring dynamic R&D spillovers across technological fields.

Methodologically, our paper is closest to Galeotti et al. (2020) and Galeotti et al. (2021).

These papers use the spectral approach and focus on the importance of the local versus the global structure of the network games. As in our paper, the higher-order eigenvalues and eigenvectors beyond the first one are important (see also Golub and Sadler (2016) for a survey of work in which the second eigenvalue is important).

A dynamical structure similar to our work appears in the context of the foundations of the gravity equation of Chaney (2018). In the dynamic network of the importers and exporters, the evolution of the contacts propagates from the local to the more distant neighbors governed by a differential equation. Chaney (2018) uses the Fourier theory to study this evolution while we use the spectral methods.

In a model of endogenous network formation of Taschereau-Dumouchel (2020), the planner optimally chooses to cluster firms. This creates a structure that is locally different from the models with the fixed networks. In particular, these densely built communities slow the propagation of the shocks. While Taschereau-Dumouchel (2020) studies these properties quantitatively, the spectral theory that we develop, albeit for a fixed network, allows to theoretically describe the properties of such regions. Similarly, in the asset-pricing applications such as Herskovic et al. (2020) understanding the determinants of the clusters that create comovement of the firms returns and volatilities may be important. In recent work, Kleinman et al. (2023) apply spectral analysis in a dynamic spatial network setting. Kikuchi et al. (2021) study a static production network using dynamic methods. They provide a general methodology of using dynamic programming by reinterpreting time as an index over decision making entities.

Our theory shows that temporary shocks to upstream sectors have outsized welfare impact. Our notion of upstreamness relates to the upstreamness measure of Antràs et al. (2012) and the distortion centrality measure of Liu (2019), as all three measures are derived from placing an increasing sequence of weights to higher-order terms in the power series of the input-output matrix.

2 Model

In this section we develop a dynamic input-output network model. Specifically, we generalize the static production network model of Acemoglu et al. (2012) by introducing dynamic adjustment costs in input-output linkages.

There is a representative consumer with exogenous labor supply $\bar{\ell}$ and N production sectors that produce from labor and intermediate inputs. The consumer has utility

$$V \equiv \int_0^\infty e^{-\rho t} \ln c(t)\, dt \tag{1}$$

where $c(t)$ is a Cobb-Douglas aggregator over sectoral goods $j = 1, \ldots, N$:

$$c(t) = \chi_c \prod_{j=1}^{N} (c_j(t))^{\beta_j}, \qquad \sum_{j=1}^{N} \beta_j = 1, \tag{2}$$

where $\chi_c \equiv \prod_{j=1}^{N} \beta_j^{-\beta_j}$ is a normalizing constant. We refer to $c(t)$ as aggregate consumption and GDP interchangeably.

At each time t, the output of production sector i satisfies

$$q_i(t) = \chi_i z_i(t) (\ell_i(t))^{\alpha_i} \prod_{j=1}^{N} (m_{ij}(t))^{\sigma_{ij}}, \qquad \sum_{j=1}^{N} \sigma_{ij} + \alpha_i = 1, \tag{3}$$

where $0 \leq \alpha_i, \sigma_{ij} \leq 1$, $\chi_i \equiv \alpha_i^{-\alpha_i} \prod_{j=1}^{N} \sigma_{ij}^{-\sigma_{ij}}$ is a normalizing constant, $z_i(t)$ is sectoral total factor productivity, $l_i(t)$ is the amount of labor used, and $m_{ij}(t)$ is the amount of the intermediate good of the sector j used in the production of the good i.

From now on, wherever it does not cause confusion, we suppress dependence on time t in the notation.

To use input m_{ij} at time t, sector i needs to buy

$$s_{ij} \equiv m_{ij} \times \exp(\delta \dot{m}_{ij} / m_{ij}) \tag{4}$$

units of input j. The term $\dot{m}_{ij} \equiv \mathrm{d}m_{ij}(t)/dt$ is the rate of expansion in the quantity of intermediate input j used by sector i. The term $\exp(\delta \dot{m}_{ij}/m_{ij})$ represents the iceberg adjustment cost that producer i has to incur when it raises the quantity of input j. The cost does not arise in steady-state: when $\dot{m}_{ij} = 0$, equation (4) implies that quantity of inputs purchased is equal to the quantity of inputs used for production ($s_{ij} = m_{ij}$). The parameter δ captures the ease of adjustment when input expands; as $\delta \to 0$, adjustment costs vanish. The formulation captures the notion that, following temporary negative shocks that destroyed input-output linkages, the recovery of these input-output linkages must be gradual. Dynamics are thus introduced through the expansion of intermediate inputs across producers, and temporary shocks therefore may have lasting impact on the economy.

The goods and labor market clearing conditions are

$$q_j = c_j + \textstyle\sum_{i=1}^{N} s_{ij} \text{ for all } j, \qquad \bar{\ell} = \textstyle\sum_{i=1}^{N} \ell_i. \tag{5}$$

For simplicity, we assume goods delivered to the consumer are not subject to adjustment costs, and neither is the use of labor across production sectors.[2]

Discussion of the Adjustment Cost Formulation The specific form of adjustment costs merits some discussion. First, adjustment costs in (4) take an exponential form. This is for analytic tractability. As we show, the exponential adjustment cost formulation implies a log-linear law of motion for the state variables (intermediate input quantities used, m_{ij}), and we can then explicitly derive the closed-form solution for the entire dynamic path of all endogenous variables in the economy. In Section 3.4.1 we extend our main result to buyer-seller-specific adjustment costs. In Section 3.4.2 we extend our main result to a more general environment environment where both the production networks and adjustment costs are non-parametric, and we show our main result holds as a first-order approximation around the steady-state.

Second, our baseline analysis focuses on the recovery path of the economy after temporary negative TFP shocks; along the recovery path, inputs always weakly expands. In essence, the negative shocks provide the initial condition of the input-output linkages, which are below

[2] These choices are made for expositional simplicity and are without loss of generality. We can always accommodate adjustment costs in the purchase of labor or the consumption good by creating a fictitious production sector that buys the consumption bundle and sells to the consumer or buys labor and sells to other producers.

the steady-state levels, and all subsequent dynamics arise due to the expansion of input-output linkages as they recover towards the steady-state. We can thus isolate the dynamics that arise due to adjustment costs and abstract away from the direct impact of TFP. In Section 3.4.3, we extend the analysis to the more general case where inputs change gradually during both the contraction and the recovery phases.

Domar Weights and Permanent TFP Shocks In what follows, we use boldface to denote vectors (lower case) and matrices (upper case). Let $\boldsymbol{\Sigma} \equiv [\sigma_{ij}]$ denote the matrix of input-output elasticities, and let $\boldsymbol{\beta}$ denote the $N \times 1$ vector of consumption elasticities. Let $\boldsymbol{\alpha}$ be the vector of sectoral value-added shares. Let $\boldsymbol{\gamma}' \equiv \boldsymbol{\beta}' (\boldsymbol{I} - \boldsymbol{\Sigma})^{-1}$ be the vector of Domar weights.

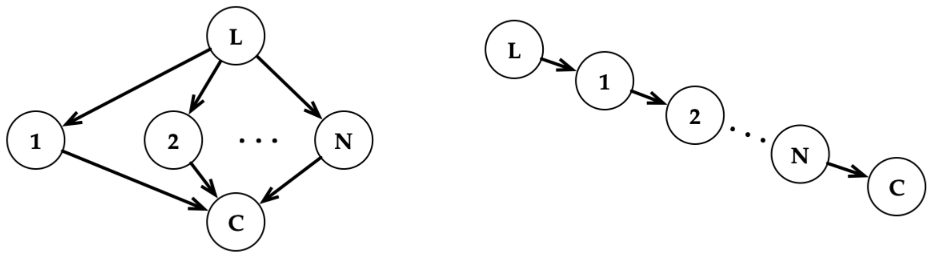
When $\delta = 0$, the economy does not feature adjustment costs, and the model becomes a repeated version of the static economy in Acemoglu et al. (2012). When $\delta > 0$, the economy features adjustment costs. However, adjustment costs are zero in steady-states (as $\dot{m}_{ij} = 0$), and the production technology in equations (2) through (5) again coincides with the static model. As we show later, the result of Acemoglu et al. (2012) holds across steady-states when shocks are permanent: the Domar weight γ_i of sector i characterizes the cross-steady-state consumption differences resulting from permanent TFP shocks.

We now turn to the analysis of temporary shocks.

3 Recovery Dynamics After Negative TFP Shocks

Consider a production network affected by temporary negative TFP shocks to some sectors. These shocks reduce sectoral production and may propagate through input-output linkages and affect the output in other sectors. We show, when production linkages take time to recover, the structure of a production network is a key determinant of its resilience to negative shocks and the time to recovery.

Figure 1. Two stylized example networks



(a) A horizontal production economy (b) A vertical production chain

To illustrate the intuition, consider two example networks. In Figure 1, panel (a) shows the network structure of a horizontal production economy. Here, labor is the only factor of production in each sector $i \in \{1, \ldots, N\}$, and sectors i does not use any of the goods $j \in \{1, \ldots, N\}$ in production. All of the goods are part of the consumption bundle c. In this setting without input-output linkages, adjustment costs are irrelevant, and traditional static analysis holds. Negative TFP shocks to larger sectors—those with higher Domar weights—have greater effects on the GDP.

Now consider panel (b), which shows the network structure of a vertical production chain. In this case, labor is the only factor of production of sector 1, and each subsequent sector i uses inputs only of the sector $i-1$. Only good N contributes to final consumption c. Consider a temporary decline in sector 1's productivity z_1. Sector 1's output declines; moreover, because sector 2 requires good 1 as inputs, the output of sector 2 declines as well, and in fact output declines in all sectors $i \in \{1, \ldots, N\}$. After the initial TFP shock dissipates and as z_1 reverts, output in sector 1 recovers immediately. However, because of adjustment costs in the recovery of input-output linkages, sectoral output for all $i \geq 2$ may stay extendedly depressed. Thus, the economy as a whole, measured by the consumption aggregator $c(t)$ (i.e., GDP), may take a long time to recover. By contrast, a temporary reduction in sector N's TFP z_N has no lasting impact on the economy, which recovers immediately after the shock dissipates. More generally, in the vertical network of panel (b), the economy recovers more slowly from negative shocks that affect relatively upstream sectors. This is despite the fact that all sectors along the supply chain may have identical Domar weights.

We now formalize this intuition and analyze the recovery of a production network after negative shocks from our dynamic perspective.

3.1 Negative TFP Shocks and Transitional Dynamics

We analyze an economy in an initial steady-state with sectoral log-productivities $\{\ln z_i\}_{i=1}^{N}$ and consider temporary, negative TFP shocks that reduce productivities in logs by $\{\widetilde{z}_i\}$ at $t = 0^-$. When the economy reaches the low-TFP steady-state, the log-GDP declines by $\boldsymbol{\gamma}'\tilde{\boldsymbol{z}}$ relative to its original level, where $\boldsymbol{\gamma}'$ is the Domar weights, consistent with Hulten (1978) and Acemoglu et al. (2012). We assume sectoral TFP reverts back to the pre-shock levels at $t = 0$, and we analyze the dynamic path of sectoral output and GDP during the recovery from $t = 0$ onwards.

Even as TFP recovers at $t = 0$, the use of intermediate inputs can only grow gradually over time and cannot jump discontinuously. Hence, sectoral output increases exactly in proportion to the TFP recovery at $t = 0$, and the total output in sector j exceeds the total quantity of good j used as production inputs. The excess output is dispensed as the adjustment costs required to expand input j for the future. With passage of time, sectors continue to expand the use of inputs, sectoral output continues to expand even though TFP is constant. Eventually the economy converges back to the initial steady-state as $t \to \infty$.

The discussion above reflects the solution to the planner's problem, which we formalize now. The primary difficulty in the analysis is as follows. Given that the quantity of intermediate inputs $\{m_{ij}\}$ cannot jump upwards, the entire matrix of input-output quantities are the state variables of the economy. Moreover, input allocations in each sector at time t has dynamic consequences and affect the output trajectory of potentially all other sectors of the economy for all future times. Nevertheless, despite the large $N \times N$ state space, and the rich network effects, we can completely solve the entire efficient transitional path in closed-form.

Formally, the planner's problem is defined as follows. The planner chooses the path of production (labor and intermediate) input allocations in order to maximize consumer welfare, taking time-0 allocations of intermediate inputs as given:

$$\text{(Planner's problem)} \qquad V\left(\{m_{ij}(0)\}\right) = \max_{\{\ell_i(\cdot), s_{ij}(\cdot)\}} \int e^{-\rho t} \textstyle\sum_j \beta_j \ln c_j \, \mathrm{d}t \tag{6}$$

$$\text{subject to } c_j = z_j \ell_j(t)^{\alpha_j} \textstyle\prod_k m_{jk}(t)^{\sigma_{jk}} - \sum_i s_{ij}(t), \tag{7}$$

$$\dot{m}_{ij}/m_{ij} = \delta^{-1}\left(\ln s_{ij} - \ln m_{ij}\right), \tag{8}$$

and the labor market clearing condition (5). Equation (7) is derived from the market clearing condition (5) for good j, and equation (8) reflects the law of motion for an expanding path of intermediate inputs under adjustment costs. A steady-state is defined by the state variables given which the planner's solution is time-invariant. We assume the time-0 allocations of intermediate inputs (i.e., the initial state of the planner's problem) correspond to allocations in the low-TFP steady-state when the negative shocks were in place at $t = 0^-$.[3]

Lemma 1. *For any initial condition $\{m_{ij}(0)\}$, the solution to the planner's problem features a constant fraction of each good j being sent to the consumer and to each input user i along the entire transitional path: $c_j(t)/q_j(t)$ and $s_{ij}(t)/q_j(t)$ are time invariant for all i, j, $t \geq 0$. The solution also features a constant labor allocation along the entire transition path, as $\ell_j(t)/\bar{\ell}$ is also time invariant for all j.*

Proof. See Appendix A.1. □

It is well known that in a static model with Cobb-Douglas preferences and production functions, the fraction of each good j sent to each input user i is invariant to sectoral TFP shocks. Lemma 1 shows that under a log-linear law of motion (8) for the state variables, which is derived from our formulation of exponential adjustment costs (4), the fraction of

[3]Choosing the low-TFP steady-state as the initial state when TFP recovers at $t = 0$ is an expositional device that enables us to study the dynamic path of the economy over $t \geq 0$ as it converges back to the initial, high-TFP steady-state. Another way to interpret this is that inputs must contract instantaneously when the shock arrives at $t = 0^-$ and can only expand gradually after the shock recedes at $t = 0$. In Section 3.4.3, we extend the analysis to the more general case where inputs change gradually during both the contraction and the recovery phase.

each good sold to each buyer is invariant along the entire transition path and is independent of the initial state variables. This is the key property that leads to tractability of our subsequent analysis despite the large state space of our model.

We now use Lemma 1 to provide a closed-form solution to the transition dynamics. Define

$$x_j(t) \equiv \ln \sum_{i=1}^{N} s_{ij}(t) - \ln \sum_{i=1}^{N} m_{ij}(t). \tag{9}$$

$x_j(t)$ is the log-ratio between the quantity of good j supplied to and used by other producers. In a steady-state, the ratio $\frac{\sum_i s_{ij}}{\sum_i m_{ij}}$ is equal to one, and $x_j = 0$ for all j. Away from a steady-state, the ratio captures the proportional adjustment costs incurred for expanding input j in production. A temporary shock always generates a common proportional decline in m_{ij} across input users i for any input j. Lemma 1 and the law of motion (8) further imply that (see Appendix A.2) both $m_{ij}(t)$ and $s_{ij}(t)$ grow at rates independent of i during the transition, so $\delta^{-1} x_j(t)$ is the rate at which all sectors expand their use of input j.

We are now ready to derive the law of motion for sectoral output and GDP.

Lemma 2. ***Laws of Motion for Sectoral Output and GDP.*** *Consider the economy recovering at time 0 from a TFP shock vector $\tilde{\boldsymbol{z}}$.*

1. *The law of motion for sectoral output vector $\boldsymbol{q}$ is*

$$\frac{\mathrm{d}\ln \boldsymbol{q}}{\mathrm{d}t} = \delta^{-1}\boldsymbol{\Sigma}\boldsymbol{x}(t), \quad \text{with initial condition } \ln \boldsymbol{q}(0) = \ln \boldsymbol{q}^{ss} - \boldsymbol{\Sigma}(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\tilde{\boldsymbol{z}}, \tag{10}$$

where $\boldsymbol{I}$ is the identity matrix and $\boldsymbol{\Sigma} \equiv [\sigma_{ij}]$ is the matrix of input-output elasticities.

2. *The law of motion for GDP is*

$$\frac{\mathrm{d}\ln c(t)}{\mathrm{d}t} = \delta^{-1}\boldsymbol{\beta}'\boldsymbol{\Sigma}\boldsymbol{x}(t), \quad \text{with initial condition } \ln c(0) = \ln c^{ss} - \boldsymbol{\beta}'\boldsymbol{\Sigma}(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\tilde{\boldsymbol{z}}. \tag{11}$$

3. *The path of the log-ratio between inputs supplied and used satisfies*

$$\frac{d\boldsymbol{x}(t)}{dt} = -\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})\boldsymbol{x}(t), \quad \text{with the initial condition } \boldsymbol{x}(0) = \tilde{\boldsymbol{z}}.$$

To understand this Lemma, first suppose the negative TFP shocks were permanent. Output declines in sectors directly affected by the shocks. Moreover, because of production linkages, output also declines in sectors that purchase inputs—directly or indirectly—from the shocked sectors. The total impact of negative shocks on sectoral output is captured by $-(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\tilde{\boldsymbol{z}}$, where the Leontief inverse $(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1} \equiv \boldsymbol{I}+\boldsymbol{\Sigma}+\boldsymbol{\Sigma}^2+\cdots$ captures the infinite rounds of higher order effects through input-output linkages. This is indeed the finding of Acemoglu et al. (2015) in the standard, static production network model.

Lemma 2 instead pertains to the persistent impact of temporary shocks. We now discuss these laws of motion's intuitions, which help illustrate the model forces. As sectoral TFP recovers, log-output directly recovers by $\tilde{\boldsymbol{z}}$; hence, at time $t=0$, sectoral output satisfies

$$\ln \boldsymbol{q}(0) = \underbrace{\ln \boldsymbol{q}^{ss}}_{\text{initial steady state}} - \underbrace{(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\tilde{\boldsymbol{z}}}_{\substack{\text{effect of permanent} \\ \text{negative shocks}}} + \underbrace{\tilde{\boldsymbol{z}}}_{\text{recovery of TFP}}.$$

The input-output linkages destroyed by the negative shocks take time to recover. Because $\delta^{-1}x_j(t) = \dot{m}_{ij}/m_{ij}$ captures the rate at which all producers expand their use of input j, the output of sector i grows at rate

$$\dot{q}_i/q_i = \delta^{-1}\sum_{j=1}^{N}\sigma_{ij}x_j(t), \tag{12}$$

which is equation (10) in scalar form. Importantly, after time $t=0$, the expansion in output is entirely due to the recovery of input-output linkages and not because of TFP changes. The impact of input j's recovery on i's output is captured by σ_{ij}, the elasticity of i's output with respect to input j. The law of motion (11) for GDP follows from the fact that a constant fraction of each good is sent to the consumer (Lemma 1); hence, log-GDP's deviation from the initial steady-state is the consumption share $\boldsymbol{\beta}$-weighted log-deviation in sectoral output.

Finally, to derive the law of motion for $x_j(t)$, the log-ratio between quantity supplied and quantity used of each input j, note that by Lemma 1, a constant share of $q_j(t)$ is sent to each sector as inputs $s_{ij}(t)$; hence, equation (9) implies

$$\frac{\mathrm{d}x_j(t)}{\mathrm{d}t} = \frac{\mathrm{d}\ln q_j(t)}{\mathrm{d}t} - \frac{\mathrm{d}\ln\left(\sum_i m_{ij}(t)\right)}{\mathrm{d}t}.$$

The first term captures the rate at which sectoral output expands; the second term is the rate at which the quantity of good j as intermediate inputs expands. By (8), the second term is equal to $\delta^{-1}x_j(t)$; hence, in the vector form,

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \delta^{-1}\boldsymbol{\Sigma}\boldsymbol{x} - \delta^{-1}\boldsymbol{x}.$$

The ODE system for $\boldsymbol{x}(t)$ has an explicit solution in terms of the matrix exponential:

$$\boldsymbol{x}(t) \quad = \quad e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})t}\tilde{\boldsymbol{z}}, \tag{13}$$

where matrix exponential for any generic matrix $\boldsymbol{M}$ is defined as $e^{\boldsymbol{M}} \equiv \sum_{k=0}^{\infty}\frac{\boldsymbol{M}^k}{k!}$.

Intuitively, immediately after TFP recovers at time 0, since inputs cannot jump discontinuously, the log-ratio between quantity supplied and quantity used as intermediate inputs for each good j is exactly captured by the magnitude of TFP recovery in sector j, i.e., $\boldsymbol{x}(0) = \tilde{\boldsymbol{z}}$. As production linkages recover over time and as the economy converges back to the initial steady-state, $\boldsymbol{x}(t)$ converges to the zero vector. The term δ modulates the rate of convergence; the system converges at a faster rate if adjustment cost δ is small. The next proposition describes the time paths or the flow of the sectoral outputs and consumption.

Proposition 1. ***Flow of Output and Consumption.*** *The flow of sectoral output satisfies*

$$\ln\boldsymbol{q}(t) \quad = \quad \ln\boldsymbol{q}^{ss} - \boldsymbol{\Sigma}(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})t}\tilde{\boldsymbol{z}}$$

and the flow of aggregate consumption satisfies

$$\ln c(t) \quad = \quad \ln c^{ss} - \boldsymbol{\beta}'\boldsymbol{\Sigma}(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})t}\tilde{\boldsymbol{z}}.$$

Proof. See Appendix A.3. □

When productivity in sector j recovers, the sector's output expands immediately, which gradually translates into the expansion of input j used in other sectors i, thereby causing i's output to expand gradually over time. The vector $\left(-(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\tilde{\boldsymbol{z}} + \tilde{\boldsymbol{z}}\right) = -\boldsymbol{\Sigma}(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\tilde{\boldsymbol{z}}$ captures the extent to which log-sectoral outputs at $t = 0$ are below their initial steady-state levels; it can be re-written as

$$-\boldsymbol{\Sigma}\left(\boldsymbol{I}-\boldsymbol{\Sigma}\right)^{-1}\tilde{\boldsymbol{z}}=-\left(\sum_{s=1}^{\infty}\boldsymbol{\Sigma}^{s}\right)\tilde{\boldsymbol{z}}$$

where each successive term in the summation captures a higher round of input-output linkages to be recovered from the initial shock. The expression $\left(-\boldsymbol{\Sigma}\left(\boldsymbol{I}-\boldsymbol{\Sigma}\right)^{-1}e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})t}\right)$ is the log-deviation in output relative to initial steady-state levels at time t; it is the continuous time analogue of the discrete partial sum $-\sum_{s=t}^{\infty}\Sigma^{s}$ that goes from $s=t$ to $s=\infty$. By varying t, the expression captures the fact that input-output linkages recover gradually, and higher rounds of linkages take longer to recover. Intuitively, a discrete sum would have implied that after t periods of recovery, the loss in output is entirely attributable to the input-output linkages higher than the t-th round, as all all prior rounds of input-output linkages have recovered. As we show below, our continuous formulation implies that every round of linkages recovers continuously as time passes, but higher rounds of linkages recover more slowly.

The rate of recovery is inversely related to δ. As $\delta\to 0$, the convergence towards the initial steady-state becomes instantaneous, as $\lim_{\delta\to 0}e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})t}=\mathbf{0}$ for any $t>0$.

More broadly, this proposition shows that the properties of the dynamical system described by the gradual adjustment of the economy are tightly related to the properties of the input-output matrix via the sequence of its powers $\boldsymbol{\Sigma}^{s}$. The parameter δ modulates the speed of adjustment.

3.2 Welfare Impact of Sectoral Shocks

We now characterize the impact of sectoral TFP shocks on consumer welfare. Let V^{ss} denote consumer welfare in the initial steady state.

Proposition 2. ***Welfare Impact of Temporary TFP Shocks.*** *Let*

$$\boldsymbol{v}'\equiv\frac{1}{\rho}\left[\boldsymbol{\beta}'\left(\boldsymbol{I}-\boldsymbol{\Sigma}\right)^{-1}-\boldsymbol{\beta}'\left(\boldsymbol{I}-\frac{\boldsymbol{\Sigma}}{1+\rho\delta}\right)^{-1}\right].\tag{14}$$

At time $t = 0$, the impact of temporary TFP shocks $\tilde{\boldsymbol{z}}$ on welfare is

$$V\left(\tilde{\boldsymbol{z}}\right) - V^{ss} = \int_0^\infty e^{-\rho s} \left(\ln c\left(s\right) - \ln c^{ss}\right) ds = -\boldsymbol{v}'\tilde{\boldsymbol{z}}.$$

Proof. See Appendix A.4. □

The vector $\boldsymbol{v}$ captures the welfare impact of temporary shocks due to the slow recovery of input-output linkages (referred to as "welfare impact" henceforth). When $\delta = 0$, recovery is instantaneous, and temporary shocks have no welfare impact. The first term, $\frac{1}{\rho}\boldsymbol{\beta}'\left(\boldsymbol{I} - \boldsymbol{\Sigma}\right)^{-1}$, is proportional to the sectoral Domar weight and captures the impact on welfare of permanent negative TFP shocks. The second term $\frac{1}{\rho}\boldsymbol{\beta}'\left(\boldsymbol{I} - \frac{\boldsymbol{\Sigma}}{1+\rho\delta}\right)^{-1}$ captures the effect of input-output recovery.

It is informative to rewrite $\boldsymbol{v}'$ as

$$\boldsymbol{v}' = \frac{1}{\rho}\boldsymbol{\beta}' \sum_{s=0}^{\infty} \left(1 - \left(1 + \rho\delta\right)^{-s}\right) \boldsymbol{\Sigma}^s \tag{15}$$

and compare the expression with sectoral Domar weights:

$$\boldsymbol{\gamma}' = \boldsymbol{\beta}' \sum_{s=0}^{\infty} \boldsymbol{\Sigma}^s. \tag{16}$$

The Domar weight captures the impact of permanent TFP shocks on steady-state consumption, and each term $\boldsymbol{\beta}'\boldsymbol{\Sigma}^s$ in the power series captures the s-th round of network effect. That is, $\boldsymbol{\beta}'$ captures the first round, direct effect of TFP on consumption, $\boldsymbol{\beta}'\boldsymbol{\Sigma}$ captures the indirect effect of sectoral TFP on other producers who supply to the consumer, and so on.

In our dynamic model, temporary shocks may have lasting effect on output and welfare after TFP recovery at $t = 0$ precisely because of higher-order linkages $\boldsymbol{\Sigma}^s$, $s > 0$. With adjustment costs ($\delta > 0$), input-output linkages are slow to recover, and $\left(1 - \left(1 + \rho\delta\right)^{-s}\right) \boldsymbol{\Sigma}^s$ captures the utility loss due to the slow recovery of the s-th order linkages. Effectively, the power series in (15) disproportionately removes the initial entries in (16) while keeping the tail entries unchanged:

$$\boldsymbol{v}' = \frac{1}{\rho}\boldsymbol{\beta}' \left[\left(1 - 1\right)\boldsymbol{\Sigma}^0 + \left(1 - \left(1 + \rho\delta\right)^{-1}\right)\boldsymbol{\Sigma}^1 + \left(1 - \left(1 + \rho\delta\right)^{-2}\right)\boldsymbol{\Sigma}^2 + ...\right]. \tag{17}$$

Note the weight on $\mathbf{\Sigma}^0$ is 0 and the weight on $\mathbf{\Sigma}^s$ converges to 1 as $s \to \infty$.[4]

We now summarize this proposition: in the presence of adjustment frictions, shocks to sectors that sell through distant linkages to the consumer are disproportionately damaging relative to the Domar weight of the affect sectors. These shocks have large and lasting impact on GDP even as sectoral TFP recovers.

Welfare Impact and Upstreamness We now show that the welfare impact measure v_i can be written as the product between sector i's Domar weight (γ_i) and a natural notion of upstreamness, which relates closely to the upstreamness measure of Antràs et al. (2012) and captures the network-adjusted distance of sectoral supply to the final consumer. Hence, temporary shocks are more damaging to the economy if they affect large sectors that are also upstream and supply disproportionate fractions of outputs to other upstream producers.

Formally, let $\eta_i \equiv v_i / \gamma_i$ be the welfare impact of a temporary shock to sector i relative to the Domar weight. We show η_i captures upstreamness.

First note that Domar weights can be written as $\boldsymbol{\gamma}' = \boldsymbol{\beta}' \sum_{s=0}^{\infty} \mathbf{\Sigma}^s$ (c.f. equation16). Antràs et al. (2012) interprets the i-th component of $\boldsymbol{\beta}'\mathbf{\Sigma}^s$ as the sales of sector i that reaches the final consumer through s-rounds of input-output linkages and defines an upstreamness measure that captures the average number of rounds it takes for sectoral output to reach the final consumer:

$$\mathrm{Up}_i = 1 \cdot \frac{\beta_i}{\gamma_i} + 2 \cdot \frac{[\boldsymbol{\beta}'\mathbf{\Sigma}]_i}{\gamma_i} + 3 \cdot \frac{[\boldsymbol{\beta}'\mathbf{\Sigma}^2]_i}{\gamma_i} + \cdots = \sum_{s=0}^{\infty} \frac{a_s \cdot [\boldsymbol{\beta}'\mathbf{\Sigma}^s]_i}{\gamma_i}, \qquad \text{with } a_s = s+1.$$

More generally, $\sum_{s=0}^{\infty} \frac{a_s \cdot [\boldsymbol{\beta}'\mathbf{\Sigma}^s]_i}{\gamma_i}$ is a measure of sector i's upstreamness for any increasing and convergent sequence $\{a_s\}_{s=0}^{\infty}$ because such a sequence up-weights the components of Domar weight that are more distant to the consumer. Our notion of upstreamness $\boldsymbol{\eta}$ can also be written in this form using the sequence $a_s = \rho^{-1}\left(1 - (1+\rho\delta)^{-s}\right)$.[5]

[4] See also Appendix A.4.1 for additional interpretations.

[5] $\boldsymbol{\eta}$ is also isomorphic to the distortion centrality of Liu (2019) in a production network with a constant wedge representing market imperfections. Liu (2019) shows that in such a setting, the distortion centrality aligns very strongly with the upstreamness measure of Antràs et al. (2012).

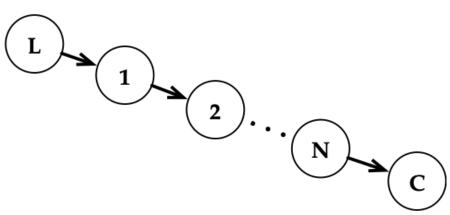
Another way interpret $\boldsymbol{\eta}$ as upstreamness is through the input-output demand matrix $\boldsymbol{\Theta}$ whose in-th entry is $\theta_{in} \equiv \sigma_{ni}\gamma_n/\gamma_i$. In the static environment of Acemoglu et al. (2012), θ_{in} is the fraction of sector i's output sold to sector n. $\boldsymbol{\eta}$ can be re-written implicitly as (see Appendix A.5) $\boldsymbol{\eta} = \frac{1}{1+\rho\delta}\boldsymbol{\Theta}\left(\boldsymbol{\eta} + \delta\right)$, or, in scalar form,

$$\eta_i = \frac{1}{1+\rho\delta}\sum_{n=1}^{N}\theta_{in}\left(\eta_n + \delta\right).$$

Hence, a sector i is upstream if it supplies disproportionately (high θ_{in}) to other relatively upstream producers (high η_n).

3.3 Vertical Example Revisited

We now revisit the examples in Figure 1. In the horizontal economy of panel (a), there are no input-output linkages; consequently, $\boldsymbol{v}$ is the zero vector, and temporary shocks have zero impact on this economy after $t = 0$. By contrast, temporary shocks may have lasting impact in the vertical economy of panel (b), with the network diagram reproduced below, along with input-output table of this economy. Sector 1 is the most upstream and sector N is the most downstream.

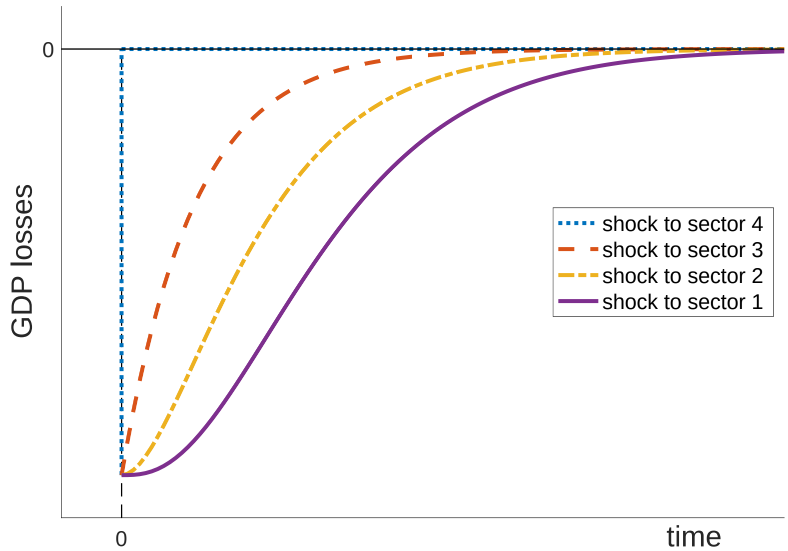


$$\boldsymbol{\Sigma} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

In this vertical economy, each successive power of the input-output matrix contains a smaller identity sub-matrix in the bottom-left and zeros otherwise, and the Leontief-inverse is a lower-triangular matrix of ones. For example, when $N = 4$,

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{\Sigma}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\Sigma}^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \left(\boldsymbol{I} - \boldsymbol{\Sigma}\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Figure 2. Time path of GDP losses from sectoral shocks in the vertical economy



By construction, the Domar weight is identically one for all sectors, $\boldsymbol{\gamma}' \equiv \boldsymbol{\beta}' (\boldsymbol{I} - \boldsymbol{\Sigma})^{-1} = \mathbf{1}'$. Permanent TFP shocks in every sector has identical impact on welfare. The welfare impact $\boldsymbol{v}$ of temporary shocks is no longer constant, however:

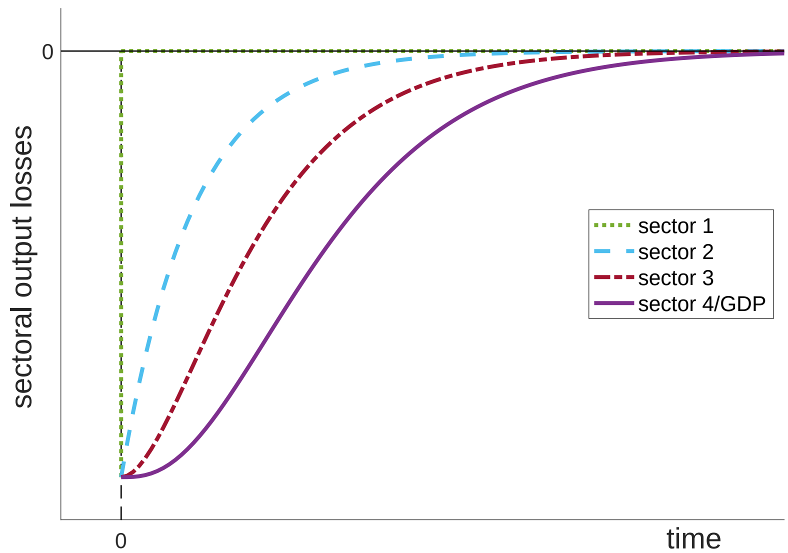
$$\boldsymbol{v}' = \rho\delta \left[\begin{array}{ccccc} 1 - (1+\rho\delta)^{-N}, & \cdots, & 1 - \left(\frac{1}{1+\rho\delta}\right)^2, & 1 - \frac{1}{1+\rho\delta}, & 0 \end{array} \right]$$

Hence, temporary shocks to an upstream sector i are more damaging than to a downstream sector $j > i$, despite all sectors having the same Domar weight.

Figure 2 shows the path of GDP for $t \geq 0$ when each sector in the vertical economy (with $N = 4$ sectors) is separately affected by a TFP shock. As the figure demonstrates, because adjustment costs compound—sector i's production needs to recover before sector $i+1$'s inputs can expand—shocks to relatively upstream sectors have long-lasting effects: the economy takes the longest time to recover from shocks to sector 1—the most upstream—and recovers instantaneously from shocks to sector 4. Consequently, $v_1 > v_2 > v_3 > v_4$ as the measure $\boldsymbol{v}$ integrates the entire path of output losses for $t \geq 0$, discounting the future at rate ρ.

Our model also has rich predictions on the recovery path of sectoral output following temporary shocks. Figure 3 shows the path of sectoral output over time when sector 1 in

Figure 3. Time path of sectoral output losses from temporary shocks to sector 1 (upstream) in the vertical economy



the vertical economy (with $N = 4$ sectors) is affected by a TFP shock. The figure shows that the more downstream a sector is from the original shock, the longer it takes for this sector's output to recover. After $t = 0$, sector 1's output recovers immediately once the TFP recovers, but the output loss lasts longer in sector 2, and even longer in sectors 3, and so on. This is because each round of input-output linkages take time to recover, and the further downstream a sector is from the original shock, the more rounds of linkages were destroyed by the initial shock and therefore need additional time to recover.

3.4 Extensions

3.4.1 Heterogeneous Adjustment Costs

In the baseline model, we assume a common adjustment cost parameter δ for all sector-pairs. We now extend our analysis to a setting with buyer-seller-pair specific adjustment costs, δ_{ij}. That is, during the recovery following a temporary TFP shock, the law of motion for input j used in sector i follows

$$\dot{m}_{ij}/m_{ij} = \delta_{ij}^{-1} \left(\ln s_{ij} - \ln m_{ij} \right),$$

replacing the law of motion (8) in the planner's problem. In Appendix A.6, we show that the impact of temporary negative TFP shocks $\tilde{\boldsymbol{z}}$ on welfare is captured by the vector

$$\boldsymbol{v}' \equiv \frac{1}{\rho}\boldsymbol{\beta}' \left[(\boldsymbol{I} - \boldsymbol{\Sigma})^{-1} - (\boldsymbol{I} - \boldsymbol{\Omega})^{-1} \right]. \tag{18}$$

Intuitively, the formula in (18), which encodes heterogeneous adjustment costs, is similar to the homogeneous adjustment cost formulation in Proposition 2, simply replacing $\frac{\boldsymbol{\Sigma}}{1+\rho\delta} \equiv \left[\frac{\sigma_{ij}}{1+\rho\delta}\right]$ by $\boldsymbol{\Omega} \equiv \left[\frac{\sigma_{ij}}{1+\rho\delta_{ij}}\right]$. The welfare impact $\boldsymbol{v}$ can be re-written as

$$\boldsymbol{v}' = \frac{1}{\rho}\left[\boldsymbol{\Sigma}^0 - \boldsymbol{\Omega}^0 + \left(\boldsymbol{\Sigma}^1 - \boldsymbol{\Omega}^1\right) + \left(\boldsymbol{\Sigma}^2 - \boldsymbol{\Omega}^2\right) + \cdots\right]$$

which also has a similar interpretation to (17).

The Domar-weight-adjusted welfare impact $\eta_i \equiv v_i/\gamma_i$ can be written as

$$\eta_i = \sum_n \frac{\delta_{ni}}{1+\rho\delta_{ni}}\theta_{in} + \sum_n \frac{\theta_{in}}{1+\rho\delta_{ni}}\eta_n \tag{19}$$

where recall $\boldsymbol{\Theta}$ is the input-output demand matrix whose in-th entry is $\theta_{in} \equiv \sigma_{ni}\gamma_n/\gamma_i$. It is a generalized version of the upstreamness measure that accounts for the heterogeneity in adjustment costs δ_{ni}. Sector i has high η if it supplies disproportionately to buyers n with high η and if these buyers are subject to substantial adjustment costs. In other words, high-η sectors are those whose output travels through many high-adjustment-cost producers, directly and indirectly, before reaching the consumer.

In general, heterogeneity in the adjustment costs do play a role in determining the size-adjusted welfare impact of temporary shocks. However, in a vertical network as in Section 3.3, a more upstream sector always has a higher η, regardless of the magnitude of the adjustment costs. Specifically, from equation (19), the size-adjusted welfare impact in the

4-sector vertical network follows

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} \propto \begin{bmatrix} 1 - \frac{1}{(1+\rho\delta_{43})(1+\rho\delta_{32})(1+\rho\delta_{21})} \\ 1 - \frac{1}{(1+\rho\delta_{43})(1+\rho\delta_{32})} \\ 1 - \frac{1}{1+\rho\delta_{43}} \\ 0 \end{bmatrix}.$$

For any adjustment cost parameters $\delta_{43}, \delta_{32}, \delta_{21} > 0$, we always have $\eta_1 > \eta_2 > \eta_3 > \eta_4$, that size-adjusted welfare impact aligns in rank order with upstreamness. This is because input flows are one directional, such that a more upstream sector's output is always subject to more compounding of adjustment costs. In Section 4.1, we show that this insight extends to the production network of the U.S. economy: the size-adjusted welfare impact of temporary shocks align very well with sectoral upstreamness, and in fact the heterogeneity in adjustment costs is quantitatively less important.

3.4.2 General Production Functions and Adjustment Costs

Our baseline model is tractable, as we obtain closed-form solution for the entire recovery path of the economy. Such tractability is achieved through a combination of log-linearity in the Cobb-Douglas production functions and in the law of motion for intermediate inputs. In this section, we extend our main welfare result (Proposition 2) to a non-parametric setting, where a version of our welfare formulas continues to hold locally (to first-order) around the initial steady-state.

Specifically, consider an economy environment in which we replace the Cobb-Douglas consumption and production functions in (2) and (3) with non-parametric aggregators that are homogeneous of degree one:

$$c(t) \equiv c\left(\{c_j(t)\}_{j=1}^N\right), \qquad q_i(t) = f_i\left(z_i(t), \ell_i(t), \{m_{ij}(t)\}_{j=1}^N\right).$$

Likewise, we consider a non-parametric adjustment cost process

$$\dot{m}_{ij} = g_{ij}(s_{ij}, m_{ij}),$$

with the requirement that $\dot{m}_{ij} = 0$ when $s_{ij} = m_{ij}$, and that $g_{ij}(\cdot)$ is locally homogeneous of degree one when $s_{ij} = m_{ij}$.[6]

Suppose we observe the economy in the initial steady-state. Let $\beta_j \equiv \partial \ln c\left(\{c_i\}_{i=1}^N\right)/\partial \ln c_j$ denote the steady-state consumption elasticity with respect to good j ($\boldsymbol{\beta} \equiv [\beta_j]$ is the corresponding vector), and $\sigma_{ij} \equiv \partial \ln q_i / \partial \ln m_{ij}$ is producer i's output elasticity with respect to input j in steady-state ($\boldsymbol{\Sigma} \equiv [\sigma_{ij}]$ is the corresponding matrix). Let $\omega_{ij}^{-1} \equiv \frac{\partial \ln g_{ij}}{\partial \ln s_{ij}}$ denote the rate at which purchased inputs s_{ij} expand the quantity of production inputs m_{ij}, again evaluated at the steady-state. Finally, let $\boldsymbol{\Omega}$ be the matrix whose ij-th entry is $\frac{\sigma_{ij}}{1+\rho\omega_{ij}}$.

In Appendix A.7, we show that, to first-order around the initial steady-state, the welfare impact of temporary shocks is

$$\left.\frac{\mathrm{d}V(\tilde{\boldsymbol{z}})}{\mathrm{d}\tilde{\boldsymbol{z}}}\right|_{\tilde{\boldsymbol{z}}\equiv 0} = -\frac{1}{\rho}\left[\boldsymbol{\beta}'\left(\boldsymbol{I}-\boldsymbol{\Sigma}\right)^{-1} - \boldsymbol{\beta}'\left(\boldsymbol{I}-\boldsymbol{\Omega}\right)^{-1}\right]. \tag{20}$$

The reduced-form object ω_{ij}, which parametrizes the rate at which purchased inputs s_{ij} expand production inputs m_{ij}, has a similar interpretation to the adjustment cost parameter in our baseline model. When ω_{ij} is common across all i, j, the formula (20) coincides with Proposition 2; when ω_{ij} is sector-pair specific, the formula coincides with the heterogeneous adjustment costs case in Section 3.4.1 if ω_{ij} is replaced by δ_{ij}.

3.4.3 When the Contraction and Expansion of Inputs Are Both Gradual

Our baseline analysis assumes that the initial conditions $\{m_{ij}(0)\}$ for the economy at $t = 0$ are provided by the low-TFP steady-state at time $t = 0^-$, and all subsequent dynamics for $t > 0$ arise due to the recovery and expansion of input-output linkages. Another way to interpret the initial condition is that inputs must contract instantaneously when the shock arrives at $t = 0^-$ and can only expand gradually after the shock recedes at $t = 0$.

In this section, we generalize the analysis and derive the output path and welfare impact when both the contraction and expansion of inputs are gradual. Specifically, we consider a

[6] That is, $\frac{\mathrm{d}g_{ij}(s,m)}{\mathrm{d}s}s + \frac{\mathrm{d}g_{ij}(s,m)}{\mathrm{d}m}m = g_{ij}(s,m)$ when $s = m$.

negative TFP shock $\widetilde{\boldsymbol{z}}$ that arrives at time 0 and recovers at time T. We characterize the efficient allocation under perfect foresight for $t \geq 0$. Note that our characterization of the planner's problem in Lemma 1 continues to hold under time-varying TFP, but the differential equations in Lemma 2 require different boundary conditions at time 0 and T, when TFP jumps discontinuously. Specifically,

1. The law of motion for sectoral output vector $\boldsymbol{q}$ is

$$\frac{\mathrm{d} \ln \boldsymbol{q}}{\mathrm{d}t} = \delta^{-1} \boldsymbol{\Sigma} \boldsymbol{x}(t)$$

   with boundary conditions $\ln \boldsymbol{q}(0) = \ln \boldsymbol{q}^{ss} - \tilde{\boldsymbol{z}}$ and $\ln \boldsymbol{q}(T) = \lim_{t \to T^-} \ln \boldsymbol{q}(t) + \tilde{\boldsymbol{z}}$.

2. The law of motion for GDP is

$$\frac{\mathrm{d} \ln c(t)}{\mathrm{d}t} = \delta^{-1} \boldsymbol{\beta}' \boldsymbol{\Sigma} \boldsymbol{x}(t)$$

   with boundary conditions $\ln c(0) = \ln c^{ss} - \boldsymbol{\beta}' \tilde{\boldsymbol{z}}$, $\ln c(T) = \lim_{t \to T^-} \ln c(t) + \boldsymbol{\beta}' \tilde{\boldsymbol{z}}$.

3. The law of motion for the log-ratio between inputs supplied and used is

$$\frac{d\boldsymbol{x}(t)}{dt} = -\delta^{-1} (\boldsymbol{I} - \boldsymbol{\Sigma}) \boldsymbol{x}(t)$$

   with boundary conditions $\boldsymbol{x}(0) = -\tilde{\boldsymbol{z}}$ and $\boldsymbol{x}(T) = \lim_{t \to T^-} \boldsymbol{x}(t) + \widetilde{\boldsymbol{z}}$.

In Appendix A.8, we show that the path of sectoral output satisfies

$$\ln \boldsymbol{q}(t) = \begin{cases} \ln q^{ss} - (\boldsymbol{I} - \boldsymbol{\Sigma})^{-1} \widetilde{\boldsymbol{z}} + \boldsymbol{\Sigma} (\boldsymbol{I} - \boldsymbol{\Sigma})^{-1} e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})t} \widetilde{\boldsymbol{z}} & t \in [0, T) \\ \ln q^{ss} - \boldsymbol{\Sigma} (\boldsymbol{I} - \boldsymbol{\Sigma})^{-1} e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})(t-T)} \left(e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})T} - \boldsymbol{I} \right) \widetilde{\boldsymbol{z}} & t \geq T \end{cases}$$

and the welfare impact is

$$V(\tilde{\boldsymbol{z}}; T) - V^{ss} = -\frac{1}{\rho} \boldsymbol{\beta}' \left(\boldsymbol{I} - \frac{1}{1 + \rho\delta} \boldsymbol{\Sigma} \right)^{-1} \left(1 - e^{-\rho T}\right) \widetilde{\boldsymbol{z}} \tag{21}$$

In this formulation, the short-run impact of the negative shock depends on the duration T of the shock; the recovery takes longer if the negative shock lasts longer.

Figure 4. Time path of GDP losses from sectoral shocks in the vertical economy

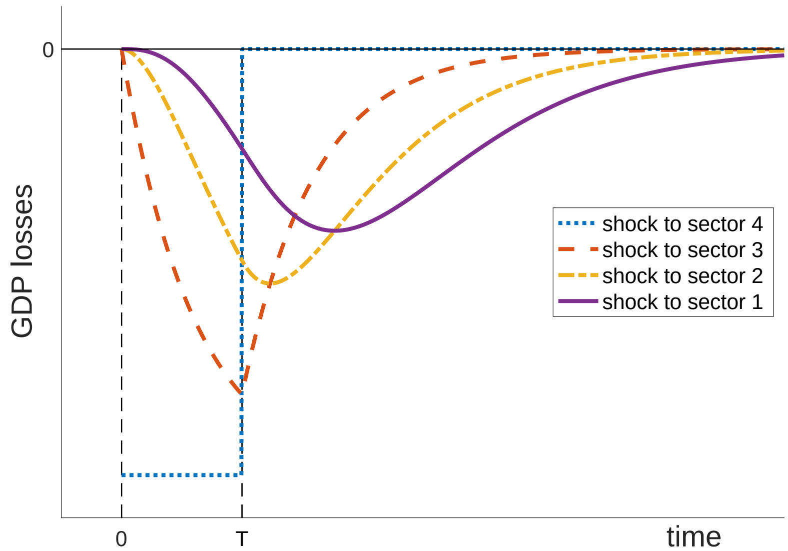


Figure 4 demonstrates the time paths of GDP losses in the vertical economy of Section 3.3 for different sectoral shocks. First consider a negative shock to the downstream sector 4. The path of GDP is depicted by the dotted blue line. Because the shock does not destroy input-output linkages—no production sector uses good 4 as inputs—the GDP moves in tandem with sector 4's TFP; the GDP collapses when the negative shock arrives at $t = 0$ and recovers completely at time T once the negative shock recedes. By contrast, consider a negative shock to the upstream sector 1. The path of GDP is depicted by the solid line. When the shock arrives at $t = 0$, the GDP does not collapse immediately—because production inputs contract gradually even as purchased inputs collapse—but instead declines gradually over time. Analogously, the GDP recovers gradually after the shock recedes at time T.

Our earlier analysis shows that, starting from the low-TFP steady-state, the economy recovers more slowly from negative shocks to the upstream sector. Figure 4 demonstrates that, conversely, starting from a high-TFP steady-state, shocks to the upstream sector takes longer to negatively affect GDP.[7]

[7]Equation (21) can also be used to analyze the welfare cost of permanent shocks, taking into account the transitional dynamics: $\lim_{T\to\infty} V\left(\tilde{\boldsymbol{z}};T\right) - V^{ss} = -\frac{1}{\rho}\boldsymbol{\beta}'\left(\boldsymbol{I} - \frac{\boldsymbol{\Sigma}}{1+\rho\delta}\right)^{-1}\tilde{\boldsymbol{z}}$. Relative to the Domar weight, which reflects the cross steady-state impact of permanent shocks and ignores the transitional dynamics, the welfare elasticity here down-weights higher rounds of network effects because they

3.4.4 Continuous TFP Recovery

Our baseline model assumes a discontinuous recovery of TFP at time $t = 0$. This section extends the analysis to the case where the TFP recovery is gradual. Specifically, suppose that after $t = 0$, the negative TFP shock recovers smoothly back to zero, with log-deviation in TFP from the initial steady-state level being $\ln z_{it} - \ln z_i \equiv e^{-\phi t}\widetilde{z}_i$. That, $\widetilde{\boldsymbol{z}}$ continues to capture the $t = 0^-$ impact of the shock, but the TFP recovers at a constant rate ϕ for $t \geq 0$ and goes back to the steady-state level as $t \to \infty$. In Appendix A.9, we analytically solves for the path of sectoral output and the welfare impact. When the matrix $((1-\phi\delta)\boldsymbol{I} - \boldsymbol{\Sigma})$ is invertible,[8] the path of sectoral output follows

$$\ln \boldsymbol{q}(t) = \ln \boldsymbol{q}^{ss} + ((1-\phi\delta)\boldsymbol{I} - \boldsymbol{\Sigma})^{-1}\left[\boldsymbol{\Sigma}\phi\delta(\boldsymbol{I} - \boldsymbol{\Sigma})^{-1} e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})t} - (1-\phi\delta)e^{-\phi t}\right]\widetilde{\boldsymbol{z}}. \tag{22}$$

The welfare impact is

$$V(\tilde{\boldsymbol{z}};\phi) - V^{ss} = \boldsymbol{\beta}'((1-\phi\delta)\boldsymbol{I} - \boldsymbol{\Sigma})^{-1}\left[\frac{\phi\delta}{\rho}\left((\boldsymbol{I}-\boldsymbol{\Sigma})^{-1} - \left(\boldsymbol{I} - \frac{\boldsymbol{\Sigma}}{1+\rho\delta}\right)^{-1}\right) - \frac{1-\phi\delta}{\rho+\phi}\boldsymbol{I}\right]\widetilde{\boldsymbol{z}}. \tag{23}$$

We make two observations. First, this result nests as special cases both our baseline model—which features discontinuous recovery of TFP at time 0—and the repeated static model with permanent TFP shocks. As $\phi \to \infty$, the speed of TFP recovery becomes instantaneous at $t = 0$, and the expressions in (22) and (23) converge to those in the baseline model (as in Propositions 1 and 2). On the other hand, as $\phi \to 0$, the TFP shock becomes permanent, and the expressions converge to those in the repeated static model, with $\ln \boldsymbol{q}(t) = \ln \boldsymbol{q}^{ss} + (\boldsymbol{I} - \boldsymbol{\Sigma})^{-1}\widetilde{\boldsymbol{z}}$ and $V(\tilde{\boldsymbol{z}};0) - V^{ss} = \frac{1}{\rho}\boldsymbol{\beta}'(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\widetilde{\boldsymbol{z}}$.

Second, equations (22) and (23) can be derived directly from our baseline model by augmenting the production network $\boldsymbol{\Sigma}$ with fictitious sectors. To see this, consider an economy

materialize more gradually in the presence of adjustment costs.

[8] We also analyze the case where $((1-\phi\delta)\boldsymbol{I} - \boldsymbol{\Sigma})$ is non-invertible in Appendix A.9. The matrix $((1-\phi\delta)\boldsymbol{I} - \boldsymbol{\Sigma})$ is non-invertible when ϕ coincides with an eigenvalue λ of $\boldsymbol{\Sigma}$. Nevertheless, the path of sectoral output and the welfare impact vary continuously in ϕ and, in case $((1-\phi\delta)\boldsymbol{I} - \boldsymbol{\Sigma})$ is non-invertible, can be derived by setting $\phi \equiv \lambda + \epsilon$ in (22) and (23) and then taking the limit as $\epsilon \to 0$, as we show in Appendix A.9.

with TFP shocks that recover exponentially at rate $\phi \equiv \delta^{-1}$ (the "original" economy). We now create an alternative economy with fictitious sectors and an discontinuous recovery of TFP. Specifically, for each sector i with production function $q_{it} = \chi_i z_{it} \ell_{it}^{\alpha_i} \prod_{j=1}^{N} m_{ijt}^{\sigma_{ij}}$ in the original economy, we create two sectors i and i_2 in the alternative economy with production functions

$$q_{it} = \chi_i z_{it} \zeta_{i_2 t}^{\alpha_i} \prod_{j=1}^{N} m_{ijt}^{\sigma_{ij}}, \quad \zeta_{i_2 t} = z_{i_2 t}^{1/\alpha_i} \ell_{i_2 t}.$$

That is, in the alternative economy, each sector i does not use labor as production inputs; instead, it uses inputs produced by sector i_2, which in turn requires labor as the only input. It is easy to see that a discontinuous recovery of $z_{i_2 t}$ generates identical impact on sectoral output $\{q_{jt}\}$ in the alternative economy as a continuous recovery of z_{it} in the original economy. This is because even though in the alternative economy, TFP $z_{i_2 t}$ recovers discontinuously, the input $\zeta_{i_2 t}$ is slow to recover due to the adjustment cost. By construction, the recovery path of $\zeta_{i_2 t}^{\alpha_i}$ in the alternative economy mimics precisely the recovery path of TFP z_{it} in the original economy; hence, we can apply our baseline results on the alternative economy to understand the impact of shocks in the original economy. Our results on heterogeneous adjustment costs in Section 3.4.1 can also be applied to study any TFP recovery rate ϕ.

4 Analysis of the U.S. Input-Output Table

We now turn to the 2012 U.S. input-output table published by the U.S. Bureau of Labor Statistics. We first provide an analysis of $\boldsymbol{v}$, the welfare impact of temporary shocks. We show $\boldsymbol{v}$ differs significantly from Domar weights $\boldsymbol{\gamma}$: while the latter is an indication of sectoral size, the former instead selects industries that are not only large but also upstream. We show that the high-dimensional input-output table—171 by 171 sectors under broad categories of agriculture, mining, manufacturing, and services[9]—has a low-dimensional, 4-factor structure

[9]13 sectors from the original 184-by-184 BLS input-output table do not use or supply any intermediate inputs and therefore do not interact with the rest of the network. These sectors are all in services, including offices of dentists, individual family services, home health care services, etc. We drop these sectors when

in terms of its susceptibility to temporary shocks: the $\boldsymbol{v}'$ vector essentially loads on only four eigenvectors of the $\boldsymbol{\Sigma}$ matrix.

4.1 Assessing the Welfare Impact $\boldsymbol{v}$ of Temporary Shocks

Calibrating Adjustment Costs To compute the welfare impact measure $\boldsymbol{v}$, we set $\rho =$ 4% as the annual discount rate. To calibrate the adjustment cost, we show in Appendix A.12 that the law of motion (8) implied by our exponential adjustment cost is first-order equivalent to a time-to-build model that is the continuous-time analogue of Long and Plosser (1983). Under this interpretation, δ corresponds to the average delay between when inputs are ordered and when they are delivered. The delay can be measured using the backlog ratio, i.e., the ratio between the stock value of unfilled orders and the flow value of goods delivered (Zarnowitz (1962), Meier (2020)). We further rely on our analysis in Section 3.4.1 and allow for input (i.e., seller) specific adjustment costs . We measure the each sector's backlog ratio using the seasonally-adjusted value from the U.S. Census M3 survey of manufacturers' shipments, inventories, and orders, which provides broad-based, monthly statistical data on economic conditions in the manufacturing sector. For each sector, we compute the average backlog ratio between the years 2010 and 2019, excluding the spike in backlogs due to the COVID-19 pandemic. We impute the backlog ratio using the sample average for input-output sectors that are not in the M3 survey. The cross-sector average backlog ratio is about 3.2 months, corresponding to $\delta_j = 0.27$ at the annual frequency. The standard deviation is 2.55 months. Sectors with the highest backlog ratios are ship and boat building (11.3 months), manufacturing of transportation equipment (10.3 months), and communications equipment manufacturing (8.2 months).

Interpreting the Welfare Impact Measure Table 1 lists the top-10 most important and least important sectors for the U.S. in terms of the welfare impact of temporary sectoral

performing the eigendecomposition.

Table 1. Welfare impact v of temporary sectoral shocks in the U.S.

10 sectors with the highest v_i	10 sectors with the smallest v_i
Real estate	Community and vocational rehabilitation services
Motor vehicle parts manufacturing	Other furniture related product manufacturing
Wholesale trade	Gambling industries (except casino hotels)
Agencies, brokerages, and other insurance related activities	Personal care services
Oil and gas extraction	Amusement parks and arcades
Management of companies and enterprises	Grantmaking, giving services, social advocacy organizations
Advertising, public relations, and related services	Food and beverage stores
Basic chemical manufacturing	Tobacco manufacturing
Employment services	Household appliance manufacturing
Petroleum and coal products manufacturing	Furniture and kitchen cabinet manufacturing

shocks. As our intuitions suggest, the most important sectors are large and supply to many other producers. The top-10 list includes very large sectors such as real estate and wholesale trade, whose sales-to-GDP ratios add to 24%. The list also includes much smaller but very upstream manufacturing sectors such as chemical and metal sectors, with sales-to-GDP ratio of only 1.9% and 1.2%, respectively. In static models, their small Domar weights would imply these sectors are unimportant; in our dynamic environment, by contrast, shocks to these sectors could create long-lasting impact because of their network positions. On the right side of the table, sectors with low welfare impact are those that are small and downstream, including many service sectors.

The left panel of Table 2 lists the top-10 sectors in terms of Domar weights $\boldsymbol{\gamma}$, which capture the welfare impact of permanent shocks. Compared with $\boldsymbol{v}$, the Domar weight $\boldsymbol{\gamma}$ captures sectoral size but disregards upstreamness. There are three sectors (wholesale trade, real estate, and petroleum and coal products manufacturing) that are on both top-10 lists, whereas seven out of ten sectors do not overlap.

The right panel of Table 2 lists the top-10 sectors in terms of size-adjusted welfare impact of temporary shocks, $\eta_i \equiv v_i/\gamma_i$. High-$\eta_i$ sectors are those whose products travel through many high-adjustment-cost producers, directly and indirectly, before reaching the consumer.

Table 2. Welfare impact of temporary v.s. permanent shocks

10 sectors with the highest Domar weight	10 sectors with the highest v_i rel. to Domar weight
Wholesale trade	Nonferrous metal (excl. aluminum) production
Real estate	Forestry
Construction	Basic chemical manufacturing
Hospitals	Agencies, brokerages, insurance related activities
Retail	Metal ore mining
Food services and drinking places	Processed steel products
Petroleum and coal products manufacturing	Support activities for agriculture and forestry
Insurance carriers	Logging
Scientific R&D	Iron and stell mills and ferroalloy manufacturing
Finance (securities, commodity contracts, funds)	Alumina and aluminum production and processing

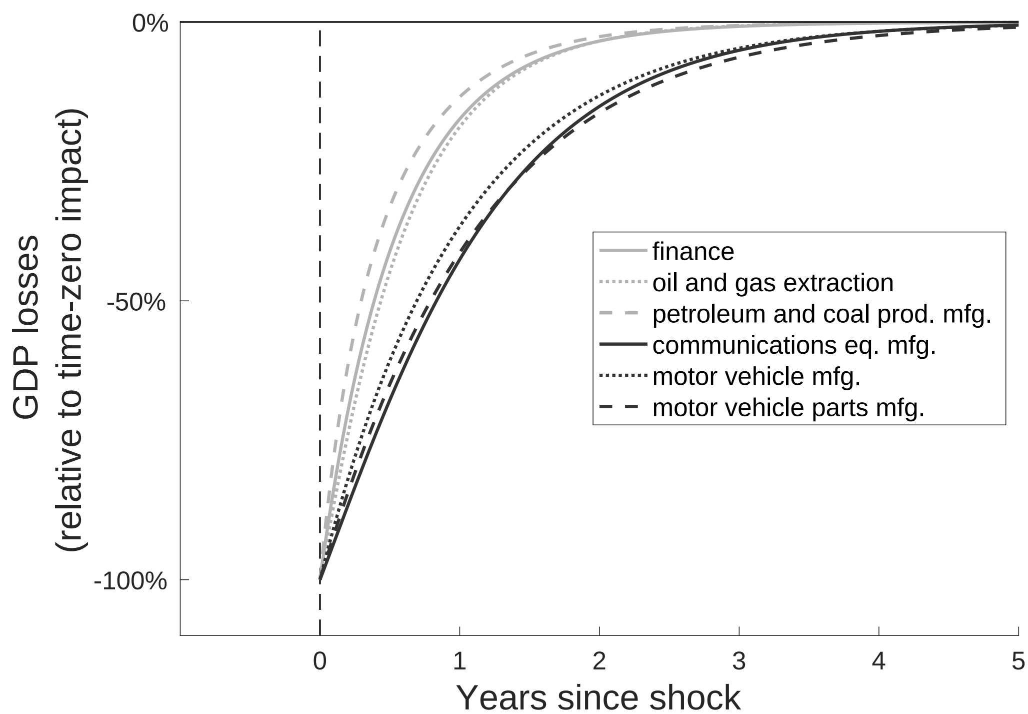
The table shows that many basic manufacturing sectors have high η's: shocks to these sectors are especially damaging relative to sectoral size because these sectors are very upstream.

Under our calibration, we find the average half-life of GDP recovery from sectoral shocks—calculated as the minimum time needed for the GDP to recover 50% of the initial loss—to be 4.8 months, with a standard deviation of 1.8 months.

Figure 5 shows the time path of GDP losses in the calibrated model from temporarily shocking six sectors, one at a time. The six sectors are separated into two groups: the first group (grey lines) consists of finance, oil and gas extraction, petroleum and coal products manufacturing; the second group (black lines) consists of the manufacturing of communication equipments, motor vehicles, and motor vehicle parts. Sectors in the first group have high Domar weights and are traditionally viewed as "important"—their (3 sectors out of 171) Domar weights add to about 12% of GDP—whereas the latter three heavy-manufacturing sectors (black lines in the figure) are relatively upstream. To control for sectoral size differences and isolate the dynamic effects, we normalize the GDP losses at time-zero to be -100% for the shock to each sector.[10]

[10]Note that the magnitude of the time-zero GDP loss from a shock to sector i is equal to the size of

Figure 5. Time path of GDP losses after sectoral shocks



The figure shows that, because adjustment costs compound, upstream sectors do experience marked real-world differences in the recovery dynamics from sectoral shocks. The half-lives of sectoral shocks to the first group average to 4.6 months, whereas the half-lives of the second group is more than twice as long and average to 9.5 months. For TFP shocks to the first group, the GDP loss one year after TFP recovery averages to 17% of the initial loss, whereas the GDP loss remains at 41% of the initial loss one year after TFP recovery of the second group. That is, conditioning on the same initial impact, the one-year GDP loss after TFP recovery from shocks to the second group is 2.4 times as large as the corresponding loss from shocks to the first group. The relative impact widens as time progresses: the three-year GDP loss after TFP recovery from shocks to the second group is 7.6 times as large as the corresponding loss from shocks to the first group.

the shock times the sector's Domar weight ($\ln c(0) - \ln c^{ss} = \gamma_i \times \widetilde{z}_i$). The two sectors, "motor vehicle manufacturing" and "oil and gas extraction", have near-identical Domar weights; hence, TFP shocks of the same magnitude to these two sectors would produce near identical short-run effects but qualitatively different dynamic effects, as shown in Figure 5.

Quantitative Evaluation of Adjustment Cost Heterogeneity Our baseline model features a homogeneous adjustment cost parameter δ, whereas our calibration so far is based on the heterogeneous adjustment cost extension in Section 3.4.1 and exploits the sectoral heterogeneity in 3M survey. We now show that the heterogeneity in adjustment costs is not quantitatively important; most variations in the welfare impact of temporary shocks arise from the network structure.

To demonstrate this, we construct a welfare impact measure v^{base} with homogeneous adjustment cost $\delta = 0.27$ for all input-pairs (calibrated to match the mean backlog ratio of 3.2 months in the 3M survey), and define $\eta_i^{base} = v_i^{base}/\gamma_i$. All cross-sector variations in the η_i^{base} arise from the network structure.

Table 3 shows the pair-wise correlations (Pearson's and Spearman's rank correlations below and above the diagonal, respectively) among η (heterogeneous adjustment cost), η^{base} (constant adjustment cost), and the upstreamness measure of Antràs et al. (2012). All three measures are near-perfectly correlated, showing that, even with heterogeneous adjustment costs, most variations in the size-adjusted welfare impact arises from the network structure.

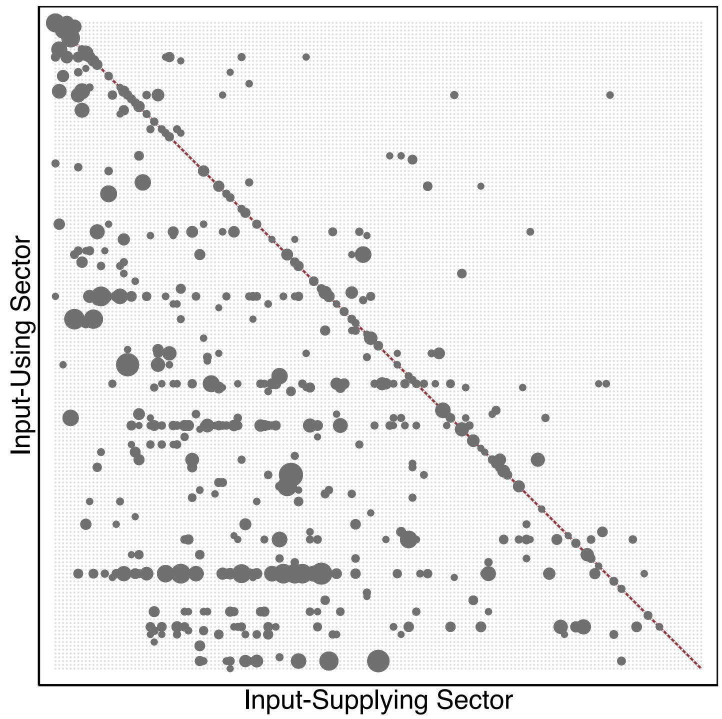
Table 3. Size-adjusted welfare impact correlates strongly with upstreamness

	η	η^{base}	Up
η	-	0.96	0.96
η^{base}	0.92	-	1.0
Up	0.92	1.0	-

Notes. This table shows the pair-wise correlations among η (heterogeneous adjustment cost), η^{base} (constant adjustment cost), and the upstreamness measure of Antràs et al. (2012). Pearson correlations are shown below the diagonal, and Spearman's rank correlation are shown above the diagonal.

High-η sectors are those whose output travels through many high-adjustment-cost producers, directly and indirectly, before reaching the consumer. Why is the heterogeneity in adjustment costs δ quantitatively unimportant? The answer lies in the structure of the U.S. input-output network. As discussed in Section 3.4.1, because adjustment costs compound through input-output linkages, a relatively upstream sector in a vertical network always has higher η, regardless of the heterogeneity in adjustment costs. Even though the U.S. economy

Figure 6. The input-output demand matrix of the U.S. economy



is not a vertical network, the "compounding" intuition still applies. Figure 6 visualizes the U.S. input-output demand matrix $\mathbf{\Theta}$, with sectors sorted by descending upstreamness. For ease of visualization, entries are drawn in proportion to θ_{ij} and are truncated below at 4%, so that only important linkages are shown.

Figure 6 shows a striking feature of the U.S. input-output network. Once sectors are sorted by upstreamness, the network appears hierarchical: sectors exhibit a clear pecking order and have highly asymmetric input-output relationships. The downstream sectors purchase heavily from the upstream ones—but not the reverse—as the matrix is dense below the diagonal and sparse above. A hierarchical structure is also evident below the diagonal, as upstream inputs are used more heavily by relatively upstream producers than by downstream producers.

In such a hierarchical network, the bulk of input flows are one directional, such that the most upstream sector's output is also subject to the most compounding of adjustment costs. This is why the size-adjusted welfare sensitivity aligns very well with sectoral upstreamness, and the heterogeneity in adjustment costs matter little. Perhaps interestingly, the hierarchical feature is not special of the U.S. production network: Liu (2019) notes the similar hierarchical feature in the input-output tables of China and South Korea, and Dhyne et al. (2022) show the Belgium production network can be well-approximated by an acyclic network, which is a network with a lower-triangular input-output demand matrix $\mathbf{\Theta}$.

In summary, most variations in the welfare impact—size-adjusted or not—arise from the network structure and not from the heterogeneity in adjustment costs. For expositional simplicity and to further isolate the role of the network structure, in subsequent analysis we adopt v^{base} as our baseline measure of the welfare impact of temporary shocks, with a single, economy-wide adjustment cost parameter $\delta = 0.27$.

4.2 Factor Structure of the U.S. Input-Output Matrix

Spectral Decomposition Our preceding analysis shows that temporary shocks to large and upstream sectors are disproportionately damaging because input-output linkages are slow to recover from these shocks. We now examine determinants of the welfare impact using the spectral point of view. Spectral analysis enables us to identify which properties of the input-output network as well as how the nature of adjustment costs matter quantitatively. Specifically, we undertake an eigendecomposition of the input-output table, $\mathbf{\Sigma} = \boldsymbol{U}\mathbf{\Lambda}\boldsymbol{W}$, where $\mathbf{\Lambda}$ is a diagonal matrix of eigenvalues $\{\lambda_k\}_{k=1}^N$ arranged in decreasing order by absolute values, and $\boldsymbol{W} = \boldsymbol{U}^{-1}$. For each eigenvalue λ_h, the h-th column of $\boldsymbol{U}$ $(\boldsymbol{u}_h)$ and the h-th row of $\boldsymbol{W}$ $(\boldsymbol{w}_h')$ are the right- and left-eigenvectors of $\mathbf{\Sigma}$, respectively, such that

$$\mathbf{\Sigma}\boldsymbol{u}_h = \lambda_h \boldsymbol{u}_h, \qquad \boldsymbol{w}_h'\mathbf{\Sigma} = \lambda_h \boldsymbol{w}_h'.$$

That is, $\boldsymbol{u}_h$ $(\boldsymbol{w}_h')$ is the vector that, when left-multiplied (right-multiplied) by $\mathbf{\Sigma}$, is proportional to itself but scaled by the corresponding eigenvalue λ_h.[11]

[11]We construct the right-eigenvectors such that the 2-norm of $\boldsymbol{u}_h$ is equal to 1 for all h.

Now consider a TFP shock whose direct impact is equal to an eigenvector $\boldsymbol{u}_h$. The second round network effect is $\boldsymbol{\Sigma}\boldsymbol{u}_h = \lambda_h \boldsymbol{u}_h$; the third round effect is $\boldsymbol{\Sigma}^2 \boldsymbol{u}_h = \lambda_h^2 \boldsymbol{u}_h$, and so on. The impact of each round of network effect decays proportionally by a factor equal to the eigenvalue λ_h. Generically, all square matrices are diagonalizable over $\mathbb{C}$, and any TFP shock vector $\tilde{\boldsymbol{z}}$ can be written as a linear combination $\{a_k\}_{k=1}^N$ of the right-eigenvectors:[12]

$$\tilde{\boldsymbol{z}} = \textstyle\sum_{k=1}^N a_k \boldsymbol{u}_k,$$

where the weights $a_k = \boldsymbol{w}_k' \widetilde{\boldsymbol{z}}$ can be recovered using the left-eigenbasis $\boldsymbol{W}$.

We now use the eigenbases $\boldsymbol{U}$ and $\boldsymbol{W}$ to further decompose the aggregate impact of sectoral shocks. In Appendix A.11, we show the Domar weight $\boldsymbol{\gamma}$ can be written as

$$\boldsymbol{\gamma}' = \boldsymbol{\beta}' \textstyle\sum_{k=1}^N \frac{1}{1-\lambda_k} \boldsymbol{u}_k \boldsymbol{w}_k' \tag{24}$$

and the welfare sensitivity $\boldsymbol{v}$ to temporary shocks can be written as

$$\boldsymbol{v}' = \boldsymbol{\beta}' \textstyle\sum_{k=1}^N \frac{\delta \lambda_k}{(1-\lambda_k)(1+\rho\delta-\lambda_k)} \boldsymbol{u}_k \boldsymbol{w}_k'. \tag{25}$$

The eigendecomposition turns the infinite-sum-of-power-series representation of $\boldsymbol{\gamma}$ and $\boldsymbol{v}$ in (16) and (15) into finite sums over N eigencomponents. To understand the implication, first consider the static model. Note that $\boldsymbol{w}_\ell' \boldsymbol{u}_k = 1$ if $\ell = k$ and is zero other wise; hence, the welfare impact of a shock profile $\widetilde{\boldsymbol{z}} = \boldsymbol{u}_k$ in the static model is captured by

$$\boldsymbol{\gamma}' \boldsymbol{u}_k = \boldsymbol{\beta}' \textstyle\sum_{\ell=1}^N \frac{1}{1-\lambda_\ell} \boldsymbol{u}_\ell \boldsymbol{w}_\ell' \boldsymbol{u}_k = \frac{1}{1-\lambda_k} \boldsymbol{\beta}' \boldsymbol{u}_k. \tag{26}$$

That is, the shock $\boldsymbol{u}_k$ affects consumption only through the k-th eigen component, with the direct effect being $\boldsymbol{\beta}' \boldsymbol{u}_k$, the s-th round indirect network effect being $\lambda_k^s \boldsymbol{\beta}' \boldsymbol{u}_k$, and a

[12] In general, these eigenvectors and eigenvalues can be complex-valued. If the direct impact of a TFP shock is the real part of a complex eigenvector $\boldsymbol{u}_h$ ($\widetilde{\boldsymbol{z}} = \mathrm{Re}(\boldsymbol{u}_h)$), then $\boldsymbol{\Sigma}^s \boldsymbol{u}_h = \mathrm{Re}(\lambda_h^s \boldsymbol{u}_h) \neq \mathrm{Re}(\lambda_h) \cdot \mathrm{Re}\left(\lambda_h^{s-1} \boldsymbol{u}_h\right)$. That is, the s-th round of network effect is captured by $\mathrm{Re}(\lambda_h^s \boldsymbol{u}_h)$, which no longer decays at a constant rate $\mathrm{Re}(\lambda_h)$. Instead, the complex eigenvalues introduce oscillatory motion as the dynamical system converges to the new steady-state. Empirically, we find that the imaginary components of the input-output table's eigenvalues are very small, implying that oscillatory effects are negligible relative to the effects that decay exponentially, so we abstract away from these complex components for expositional brevity.

cumulative effect of $\sum_{s=0}^{\infty} \lambda_k^s \boldsymbol{\beta}' \boldsymbol{u}_k = \frac{1}{1-\lambda_k} \boldsymbol{\beta}' \boldsymbol{u}_k$. By contrast, $\boldsymbol{v}' \boldsymbol{u}_k$ can be re-written as

$$\boldsymbol{v}' \boldsymbol{u}_k = \frac{1}{\rho} \boldsymbol{\beta}' \boldsymbol{u}_k \left(\sum_{s=0}^{\infty} \left(1 - (1+\rho\delta)^{-s}\right) \lambda_k^s \right) = \frac{\delta \lambda_k}{(1-\lambda_k)(1+\rho\delta-\lambda_k)} \boldsymbol{\beta}' \boldsymbol{u}_k. \tag{27}$$

The term $\left(1 - (1+\rho\delta)^{-s}\right)$ assigns zero weight to the direct effect of the shock ($s = 0$) and an increasing sequence of weights to higher-order network effects.

The U.S. Input-Output Matrix We now describe the next empirical results of the paper: the welfare impact $\boldsymbol{v}$ of temporary shocks in the U.S. can be well-approximated in a low-dimensional factor representation. Four eigencomponents (out of $N = 171$) capture most of the variation in $\boldsymbol{v}$, suggesting that these eigenvectors are all that is needed to summarize the welfare impact of temporary shocks in the U.S. input-output network through the lens of our dynamic model. This is not the case for permanent shocks in our model (or shocks in static models): the Domar weight does not have a low-dimensional representation.

Specifically, let $\boldsymbol{v}'_{(h)} \equiv \delta \boldsymbol{\beta}' \sum_{k=1}^{h} \frac{\lambda_k}{(1-\lambda_k)(1+\rho\delta-\lambda_k)} \boldsymbol{u}_k \boldsymbol{w}'_k$ denote the partial sum of the first h eigencomponents in (25), with $\boldsymbol{v} = \boldsymbol{v}_{(171)}$. $\boldsymbol{v}_{(h)}$ captures the welfare impact of sectoral shocks through the first h eigencomponents. We show that $\boldsymbol{v}_{(4)}$ approximates $\boldsymbol{v}$ very well.

Table 4 shows the regression of $\boldsymbol{v}_{(h)}$ on $\boldsymbol{v}$ for $h \in \{1, \ldots, 6\}$ and reports the slope coefficients and adjusted R^2. The first 3 eigenvectors capture 76% of the variation in $\boldsymbol{v}$; the first 4 eigenvectors capture 95% of the variation. That is, most of the welfare impact of any sectoral shock can by explained by the loading of the shock on the first four eigenvectors. Appendix Figure 7 scatterplots $\boldsymbol{v}_{(h)}$ against $\boldsymbol{v}$ for $h \leq 6$. The figure shows that $\boldsymbol{v}'_{(4)}$ approximates $\boldsymbol{v}'$ very well, and additional eigencomponents do not seem to significantly improve the fit.

Table 4. Regression of $\boldsymbol{v}_{(h)}$ on $\boldsymbol{v}$

h	1	2	3	4	5	6
slope	0.53	0.82	1.01	0.97	0.97	0.96
R^2	0.39	0.58	0.76	0.95	0.96	0.94

Table 5 conducts the analogous exercise for the Domar weight. We compute the partial sum $\boldsymbol{\gamma}'_{(h)} = \boldsymbol{\beta}' \sum_{k=1}^{h} \frac{1}{1-\lambda_k} \boldsymbol{u}_k \boldsymbol{w}'_k$ as the first h eigencomponents of Domar weights (with the actual Domar weight $\boldsymbol{\gamma} = \boldsymbol{\gamma}_{(171)}$ c.f. equation 24). Table 5 shows that the first four eigencomponents explain almost no variation in Domar weights, with an R^2 close to zero. In fact, the R^2 remains close to zero even with 160 eigencomponents of the largest eigenvalues, and almost all eigencomponents are needed to explain variations in Domar weights. Appendix Figure 8 shows that Domar weights are almost orthogonal to approximations of up to $h \leq 6$ dimensions.

Table 5. Regression of $\boldsymbol{\gamma}_{(h)}$ on $\boldsymbol{\gamma}$

h	1	2	3	4	$\cdots$	160	170
slope	0.02	0.03	0.04	0.04	$\cdots$	0.47	1.14
R^2	0.00	0.00	0.00	0.00	$\cdots$	0.03	0.62

These results imply that, to understand the dynamic impact of temporary shocks, only four eigenvectors are needed to approximate the impact of all (171 dimensional) network effects. By contrast, such a low-dimensional representation does not exists for the Domar weights; hence, to understand the impact of permanent shocks, one needs information of the entire input-output table.[13]

Equations (24) and (25) give intuition for this result. Intuitively, the Domar weight represents an non-discounted summation through the infinite rounds of network effects ($I + \boldsymbol{\Sigma} + \boldsymbol{\Sigma}^2 + \cdots$); the initial rounds are equally important as the later rounds. By contrast, the welfare impact of temporary shocks represents a weighted summation ($(1-1)\, I + \left(1 - \frac{1}{1+\rho\delta}\right) \boldsymbol{\Sigma} + \left(1 - \frac{1}{(1+\rho\delta)^2}\right) \boldsymbol{\Sigma}^2 + \cdots$) with increasing weights on later rounds of network effects, as they represent long-lasting damages due to adjustment costs, and initial rounds are heavily under-weighted. An eigencomponent with a small eigenvalue—meaning it represents a dimension of network effect that decays quickly over iterative powers of the input-output matrix—has little impact on welfare if the shocks are temporary (for the component's lack of

[13]See Appendix Figure 9 for an additional visualization that contrasts the contribution of each eigencomponent to the Domar weight $\boldsymbol{\gamma}$ and to $\boldsymbol{v}$.

impact at higher rounds) and but is potentially very important if the shocks are permanent.

Specifically, consider two distinct eigenvectors $\boldsymbol{u}_k$ and $\boldsymbol{u}_\ell$ with $|\lambda_k| < |\lambda_\ell|$. Let $|\boldsymbol{\gamma}'\boldsymbol{u}_k| \Big/ |\boldsymbol{\gamma}'\boldsymbol{u}_\ell|$ denote the relative importance of these components in the static model. In our dynamic model, their relative importance is (as implied by 26 and 27)

$$\frac{|\boldsymbol{v}'\boldsymbol{u}_k|}{|\boldsymbol{v}'\boldsymbol{u}_\ell|} = \underbrace{\frac{|\lambda_k|\,|1+\rho\delta-\lambda_\ell|}{|\lambda_\ell|\,|1+\rho\delta-\lambda_k|}}_{<1} \times \frac{|\boldsymbol{\gamma}'\boldsymbol{u}_k|}{|\boldsymbol{\gamma}'\boldsymbol{u}_\ell|}. \tag{28}$$

That is, relative to the static model, our dynamic model up-weights the relative importance of slow-decay eigenvectors (those with greater eigenvalues) and, conversely, down-weights fast-decay eigenvectors (those with lower eigenvalues). These differences could be significant: as we show below, for the U.S. economy, the dominant eigenvalue is $\lambda_1 \approx 0.54$, and the 100th is $\lambda_{100} \approx 0.03$. Hence, relative to the dominant eigencomponent, the 100-th component is at least 18 times more important in the static model than in our dynamic model. This qualitative fact, that fast-decay eigencomponents are significantly less important in the dynamic model than in the static model, holds regardless of the parametrization of ρ and δ and is the reason behind the low-dimensional representation of input-output tables in our dynamic economy.

Which Sectors Do the First Four Eigenvectors Represent? We now describe the first four eigenvectors.[14] The first eigenvector $\boldsymbol{u}_1$ represents the heavy manufacturing sectors, including metal products, foundries, forging and stamping, and as well as the production of boiler tanks, machinery, electrical and transportation equipment. This eigenvector captures the vector of TFP shocks under which the economic damage occurs disproportionately through higher rounds of network effects.

The second eigenvector $\boldsymbol{u}_2$ negatively correlates with the first (Pearson correlation coefficient of -0.59). $\boldsymbol{u}_2$ represents shocks to three groups of industries. First and most notably, $\boldsymbol{u}_2$ has large positive entries for the two sectors relating to agencies, brokerages, and insur-

[14] Appendix Figure 10 visualizes the first four eigenvectors.

ance. Second, $\boldsymbol{u}_2$ has positive entries for the manufacturing of consumer goods including food, textile, paper products, and furniture. Third, $\boldsymbol{u}_2$ has negative entries on the heavy manufacturing industries, partly neutralizing the first eigenvector.

The third eigenvector $\boldsymbol{u}_3$ correlates positively with $\boldsymbol{u}_2$—correlation coefficient 0.36—by having positive entries on the manufacturing of consumer goods. In addition, $\boldsymbol{u}_3$ also includes sectors that manufacture chemicals, plastic, and rubber products.

The fourth eigenvector has close-to-zero correlations with the previous three eigenvectors. The new sector picked up by $\boldsymbol{u}_4$ is radio and television broadcasting; in addition, $\boldsymbol{u}_4$ also has negative entries on the manufacturing of chemicals, plastic, and rubber products, partly neutralizing $\boldsymbol{u}_3$.

Altogether, the eigenvectors $\boldsymbol{u}_1$ through $\boldsymbol{u}_4$ form a 4-dimensional subspace of the 171-dimensional vector space in which the U.S. input-output table lies.

5 Conclusion

We develop a tractable dynamic model of input-output networks that incorporates adjustment costs of changing inputs. The model is solved in closed form and provides analytical insights into the gradual recovery from temporary productivity shocks and their welfare implications. The model demonstrates the critical role of input-output network structure in understanding shock propagation and its welfare consequences. Our findings also reveal the importance of upstream sectors in shock recovery and emphasizes the low-dimensional representation of the network for capturing the welfare impact of sectoral shocks.

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Appendix

A Proofs and Additional Theoretical Results

A.1 Proof to Lemma 1

Consider the planner's problem in (6). We use the following change of variables: let $v_{ij}(t) \equiv s_{ij}(t)/q_j(t)$ denote the fraction of good j sent to sector i at time t. Then consumption of good j is $c_j(t) = \left(1 - \sum_i v_{ij}(t)\right) q_j(t)$. Taking logs of the production function in (3) and recognizing that TFP is constant during the recovery path, we can equivalently write the planner's problem as

$$V\left(\{\ln m_{ij}(0)\}\right) = \max_{\{\ell_j(\cdot), v_{ij}(\cdot)\}} \int e^{-\rho t} \textstyle\sum_j \beta_j \left(\alpha_j \ln \ell_j(t) + \displaystyle\sum_k \sigma_{jk} \ln m_{jk}(t) + \ln\left(1 - \sum_i v_{ij}(t)\right)\right) \mathrm{d}t$$

$$\text{s.t.} \quad \frac{\mathrm{d} \ln m_{ij}(t)}{\mathrm{d}t} = \delta^{-1}\left(\ln v_{ij}(t) + \ln z_j + \alpha_j \ln \ell_j(t) + \sum_k \sigma_{jk} \ln m_{jk}(t) - \ln m_{ij}(t)\right)$$

$$\sum_j \ell_j(t) = \bar{\ell}$$

In what follows, we omit the time argument whenever the context is clear. Form the current-value Hamiltonian, where for notational simplicity we suppress the dependence on time for the control, state, and co-state variables:

$$\begin{aligned} H\left(\{\ell_j\}, \{\ln m_{jk}\}, \{v_{ij}\}, t\right) \quad = \quad & \textstyle\sum_j \beta_j \left(\alpha_j \ln \ell_j + \displaystyle\sum_k \sigma_{jk} \ln m_{jk} + \ln\left(1 - \sum_i v_{ij}\right)\right) \\ & + \delta^{-1} \sum_{ij} \mu_{ij} \left(\ln v_{ij} + \ln z_j + \alpha_j \ln \ell_j + \sum_k \sigma_{jk} \ln m_{jk} - \ln m_{ij}\right) \\ & + \lambda \left[\bar{\ell} - \sum_j \ell_j\right]. \end{aligned}$$

By the maximum principle,

$$H_{\ell_j} = 0 \iff \frac{\alpha_j \left(\beta_j + \delta^{-1} \sum_i \mu_{ij}\right)}{\ell_j} = \lambda \quad \text{for all } j. \tag{29}$$

$$H_{v_{ij}} = 0 \iff \frac{\beta_j}{1-\sum_i v_{ij}} = \frac{\mu_{ij}\delta^{-1}}{v_{ij}} \tag{30}$$

$$H_{\ln m_{jk}} = \rho\mu_{jk} - \dot{\mu}_{jk} \iff \beta_j\sigma_{jk} - \mu_{jk}\delta^{-1} + \delta^{-1}\sum_i \mu_{ij}\sigma_{jk} = \rho\mu_{jk} - \dot{\mu}_{jk} \tag{31}$$

We now show that the transversality condition $\lim_{t\to\infty} e^{-\rho t} H\left(\{\ell_j\}, \{\ln m_{jk}\}, \{v_{ij}\}, t\right) = 0$ implies $\dot{\mu}_{jk}(t) = 0$ for all j, k, t; the Lemma is then immediate, i.e., v_{ij} and ℓ_j are time-invariant for all i, j.

To show $\dot{\mu}_{jk}(t) = 0$ for all j, k, t, we proceed in two steps. In the first step, we define $\xi_k \equiv \sum_j \mu_{jk}$ and show $\dot{\xi}_k(t) = 0$ for all k, t. We then show $\dot{\mu}_{jk}(t) = 0$.

Using the definition of ξ_k, we sum both sides of equation (31) across j and get

$$\sum_j \beta_j\sigma_{jk} - \xi_k\delta^{-1} + \delta^{-1}\sum_j \xi_j\sigma_{jk} = \rho\xi_k - \dot{\xi}_k$$

In vector notation,

$$\dot{\boldsymbol{\xi}} = \delta^{-1}\left(\boldsymbol{I} - \boldsymbol{\Sigma}' + \rho\delta\right)\boldsymbol{\xi} - \boldsymbol{\Sigma}'\boldsymbol{\beta}$$

Thus

$$\begin{aligned}
\boldsymbol{\xi}(t) &= e^{\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma}'+\rho\delta)t}\boldsymbol{\xi}_0 - \int_0^t e^{\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma}'+\rho\delta)(t-s)}ds\boldsymbol{\Sigma}'\boldsymbol{\beta} \\
&= e^{\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma}'+\rho\delta)t}\boldsymbol{\xi}_0 - \delta\left(e^{\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma}'+\rho\delta)t} - \boldsymbol{I}\right)\left(\boldsymbol{I} - \boldsymbol{\Sigma}' + \rho\delta\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\beta}
\end{aligned}$$

Transversality implies

$$\begin{aligned}
\boldsymbol{0} &= \lim_{t\to\infty} e^{-\rho t}\boldsymbol{\xi}(t) \\
&= \lim_{t\to\infty} e^{\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma}')t}\left[\boldsymbol{\xi}_0 - \delta\left(\boldsymbol{I} - \boldsymbol{\Sigma}' + \rho\delta\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\beta}\right]
\end{aligned}$$

which is true only if

$$\boldsymbol{\xi}_0 = \delta\left(\boldsymbol{I} - \boldsymbol{\Sigma}' + \rho\delta\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\beta}$$

thereby implying $\dot{\boldsymbol{\xi}}(t) = 0$ for all t.

We now show $\dot{\mu}_{ij}(t) = 0$ for all t. Note equation (31) can be written as

$$\beta_j\sigma_{jk} - \mu_{jk}\delta^{-1} + \delta^{-1}\xi_j\sigma_{jk} = \rho\mu_{jk} - \dot{\mu}_{jk}$$

or in matrix form

$$\dot{\boldsymbol{\mu}} = \left(\delta^{-1} + \rho\right)\boldsymbol{\mu} - Diag\left(\boldsymbol{\beta} + \delta^{-1}\boldsymbol{\xi}\right)\boldsymbol{\Sigma}$$

Following the same strategy above, we can integrate and write

$$\boldsymbol{\mu}(t) = e^{\left(\delta^{-1}+\rho\right)t}\boldsymbol{\mu}_0 - \left(\delta^{-1} + \rho\right)^{-1}\left(e^{\left(\delta^{-1}+\rho\right)t} - 1\right)Diag\left(\boldsymbol{\beta} + \delta^{-1}\boldsymbol{\xi}\right)\boldsymbol{\Sigma}$$

The transversality condition $\mathbf{0} = \lim_{t\to\infty} e^{-\rho t}\boldsymbol{\mu}(t)$ again implies

$$\left(\delta^{-1}+\rho\right)\boldsymbol{\mu}(0) = Diag\left(\boldsymbol{\beta}+\delta^{-1}\boldsymbol{\xi}\right)\boldsymbol{\Sigma}$$

and thus $\dot{\boldsymbol{\mu}}(t) = \mathbf{0}$ for all t.

We now characterize the model further using the planner's solution and provide an alternative proof to our main result, Proposition 2. This characterization will be useful in Section A.7 below, where we consider non-parametric production and adjustment cost functions. First, note that μ_{ij} is the marginal value of an additional unit of m_{ij} in place at time 0. Under the log-linear baseline model, μ_{ij} is time-invariant. We can actually solve for μ_{ij} in closed-form. Using (31), note

$$\delta\beta_j + \sum_i \mu_{ij} = (1+\rho\delta)\,\mu_{jk}/\sigma_{jk}$$

Which shows μ_{jk}/σ_{jk} is independent of k. Define $\zeta_j \equiv \mu_{jk}/\sigma_{jk}$; then

$$\delta\beta_j + \sum_i \zeta_i\sigma_{ij} = (1+\rho\delta)\,\zeta_j$$

In matrix notation,

$$(1+\rho\delta)\,\boldsymbol{\zeta}' = \delta\boldsymbol{\beta}' + \boldsymbol{\zeta}'\boldsymbol{\Sigma} \iff \boldsymbol{\zeta}' = \delta\boldsymbol{\beta}'\left((1+\rho\delta)\,\boldsymbol{I}-\boldsymbol{\Sigma}\right)^{-1}.$$

To figure out the welfare impact of a temporary TFP shock, note that the impact of shock $\widetilde{\boldsymbol{z}}$ on state variables $\ln m_{ij}$ at time 0 is the j-th entry of vector $(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\widetilde{\boldsymbol{z}}$; that is, the TFP shock affects m_{ij} to the same proportion as it affects the output of sector j. Hence, the impact of $\widetilde{\boldsymbol{z}}$ on welfare is

$$\begin{aligned}
V(\widetilde{\boldsymbol{z}}) - V^{ss} &= -\sum_{ij}\mu_{ij}\left[(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\widetilde{\boldsymbol{z}}\right]_j \\
&= -\sum_{ij}\zeta_i\sigma_{ij}\left[(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\widetilde{\boldsymbol{z}}\right]_j \\
&= -\boldsymbol{\zeta}'\boldsymbol{\Sigma}(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\widetilde{\boldsymbol{z}} \\
&= -\delta\boldsymbol{\beta}'\left((1+\rho\delta)\,\boldsymbol{I}-\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\widetilde{\boldsymbol{z}} \\
&= -\frac{1}{\rho}\boldsymbol{\beta}'\left[(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1} - \left((1+\rho\delta)\,\boldsymbol{I}-\boldsymbol{\Sigma}\right)^{-1}\right]\boldsymbol{\Sigma}\widetilde{\boldsymbol{z}} \\
&= -\frac{1}{\rho}\boldsymbol{\beta}'\left[(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1} - \left(\boldsymbol{I}-\frac{\boldsymbol{\Sigma}}{1+\rho\delta}\right)^{-1}\right]\widetilde{\boldsymbol{z}},
\end{aligned}$$

as in Proposition 2.

A.2 Rate of Input Recovery

Lemma 3. *Consider a temporary TFP shock* $\widetilde{\boldsymbol{z}}$ *that recovers at time* 0*. In the planner's solution, along the entire transition path* $t \geq 0$ *and for every input* j*,* $\frac{\mathrm{d} \ln m_{ij}(t)}{\mathrm{d}t} = \delta^{-1} x_j(t)$ *for all* i*.*

Proof. We need to show $x_j(t) = \ln s_{ij}(t) - \ln m_{ij}(t)$ for all i. First note that the time-0 impact of a negative TFP shock on the state variables (the quantity of intermediate inputs $\{m_{ij}\}$) is proportional to the impact on the output of each intermediate supplier j. That is, $\ln m_{ij}(0) - \ln m_{ij}^{ss}$ is the same across all input-buyers i. Second, because $s_{ij}(t) / q_{ij}(t)$ is time-invariant (Lemma 1), we know $\ln s_{ij}(t) - \ln s_{ij}^{ss}$ is the same for all i. Integrating the law of motion (8), these two facts imply $\ln s_{ij}(t) - \ln m_{ij}(t)$ is independent of i for all t, and that $\frac{m_{ij}(t)}{\sum_{i'} m_{i'j}(t)} = \frac{s_{ij}(t)}{\sum_{i'} s_{i'j}(t)}$ and both sides are time-invariant, implying $\ln s_{ij}(t) - \ln m_{ij}(t) = \ln \sum_{i'} s_{i'j}(t) - \ln \sum_{i'} m_{i'j}(t)$ for all i and t, establishing the Lemma. □

A.3 Proof to Proposition 1

We have

$$\begin{aligned}
\ln \boldsymbol{q}(t) &= \ln \boldsymbol{q}(0) + \delta^{-1} \boldsymbol{\Sigma} \int_0^t \boldsymbol{x}(s)\, ds \\
&= \ln \boldsymbol{q}(0) + \delta^{-1} \boldsymbol{\Sigma} \left[\int_0^t e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})s} ds \right] \tilde{\boldsymbol{z}} \\
&= \underbrace{\ln \boldsymbol{q}^{ss} - (\boldsymbol{I} - \boldsymbol{\Sigma})^{-1} \tilde{\boldsymbol{z}} + \tilde{\boldsymbol{z}}}_{\boldsymbol{q}(0)} + \boldsymbol{\Sigma} (\boldsymbol{I} - \boldsymbol{\Sigma})^{-1} \left(\boldsymbol{I} - e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})t} \right) \tilde{\boldsymbol{z}} \\
&= \ln \boldsymbol{q}^{ss} - \boldsymbol{\Sigma} (\boldsymbol{I} - \boldsymbol{\Sigma})^{-1} e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})t} \tilde{\boldsymbol{z}}.
\end{aligned}$$

The expression for $c(t)$ is derived analogously.

A.4 Proof to Proposition 2

We have

$$
\begin{aligned}
V(\tilde{\boldsymbol{z}}) - V_0^{ss} &= \int_0^\infty e^{-\rho s} \left(\ln c(s) - \ln c_0^{ss}\right) ds \\
&= -\boldsymbol{\beta}' \boldsymbol{\Sigma} \left(\boldsymbol{I} - \boldsymbol{\Sigma}\right)^{-1} \int_0^\infty e^{-\delta^{-1}((1+\rho\delta)\boldsymbol{I} - \boldsymbol{\Sigma})t} dt \tilde{\boldsymbol{z}} \\
&= -\delta \boldsymbol{\beta}' \boldsymbol{\Sigma} \left(\boldsymbol{I} - \boldsymbol{\Sigma}\right)^{-1} \left((1+\rho\delta)\boldsymbol{I} - \boldsymbol{\Sigma}\right)^{-1} \tilde{\boldsymbol{z}} \\
&= -\frac{1}{\rho} \boldsymbol{\beta}' \boldsymbol{\Sigma} \left[\left(\boldsymbol{I} - \boldsymbol{\Sigma}\right)^{-1} - \left((1+\rho\delta)\boldsymbol{I} - \boldsymbol{\Sigma}\right)^{-1}\right] \tilde{\boldsymbol{z}} \\
&= -\frac{1}{\rho} \left[\boldsymbol{\beta}' \left(\boldsymbol{I} - \boldsymbol{\Sigma}\right)^{-1} - \boldsymbol{\beta}' \left(\boldsymbol{I} - \frac{\boldsymbol{\Sigma}}{1+\rho\delta}\right)^{-1}\right] \tilde{\boldsymbol{z}}.
\end{aligned}
$$

A.4.1 Alpha centrality and global versus local influence

The welfare impact measure $\boldsymbol{v}'$ of temporary shocks is connected to the notion of alpha centrality in a network represented by the input-output matrix. The alpha centrality for $\alpha \in (0, 1]$ is defined as:

$$\boldsymbol{\iota}_\alpha' \equiv \boldsymbol{\beta}'(\boldsymbol{I} - \alpha\boldsymbol{\Sigma})^{-1}.$$

Intuitively, this is a centrality measure where a parameter α is used to weigh the higher order input-output linkages, represented by the powers of the matrix $\boldsymbol{\Sigma}$:

$$\boldsymbol{\iota}_\alpha' \equiv \boldsymbol{\beta}' \left[\boldsymbol{\Sigma}^0 + \alpha\boldsymbol{\Sigma}^1 + \alpha^2\boldsymbol{\Sigma}^2 + ...\right].$$

The i-th entry in $\boldsymbol{\beta}'\boldsymbol{\Sigma}^s$ captures the component of sector i's Domar weight that is attributed to s rounds of linkages.

A related way to think about centrality is in terms of a random walk on the network, where $\boldsymbol{\Sigma}_{ij}$ is the probability of reaching j from i in one walk. The ij-th entry in $\boldsymbol{\Sigma}^s$ then measures the probability of reaching j from i in the walks of length s. As parameter ($\alpha \leq 1$) decreases, shorter walks become more important, and local influences carry higher significance. When α increases, longer walks become more important, and global influences carry higher significance. In the limit case as $\alpha \to 1$, the walks of any length carry identical weights, and the alpha centrality measure becomes the Domar weight. In this sense, alpha centrality tunes between rankings based on short walks (local influence) and those based on long walks (global influence).

The welfare impact measure $\boldsymbol{v}'$ is thus proportional to the difference in the alpha centralities $\boldsymbol{\iota}_{\alpha_1}' - \boldsymbol{\iota}_{\alpha_2}'$, where $\alpha_1 = 1$ and $\alpha_2 = (1+\rho\delta)^{-1}$. It corresponds to a generalized version of alpha centrality:

$$\tilde{\boldsymbol{\iota}}' \equiv \boldsymbol{\beta}' \left[a_0\boldsymbol{\Sigma}^0 + a_1\boldsymbol{\Sigma}^1 + a_2\boldsymbol{\Sigma}^2 + ...\right],$$

for some sequence $\{a_0, a_1 \dots\}$. Assuming that such weighted power series converge, this measure weights the walks of length k with the parameter a_k. In the case of alpha centrality with $\alpha < 1$, $a_k = \alpha^k$ and is geometrically decreasing from $a_0 = 1$ and $a_\infty = 0$. The welfare measure $\boldsymbol{v'}$ is a generalized alpha centrality with $a_k = 1 - \alpha_2^k$ and thus increasing between $a_0 = 0$ and $a_\infty = 1$. Because the measure $\boldsymbol{v}$' captures the welfare losses due to slow recovery of inputs, it relatively prioritizes the longer walks or higher order input output linkages and thus the global over local influences.

The term $(1+\rho\delta)^{-1}$ in (14) also defines a one-parameter family of economies that can be thought of as a multi-scale representation of the static input output matrix. Specifically, the speed of adjustment and the discount factor of the agent determine the scale—the relative importance of the higher-order links and thus the importance of the global versus local structures.

A.5 Welfare Impact and Upstreamness

Let $r \equiv \frac{1}{1+\rho\delta}$ and $\boldsymbol{a'} \equiv \boldsymbol{\beta}' (\boldsymbol{I} - r\boldsymbol{\Sigma})^{-1}$. We have

$$\rho\eta_j = 1 - a_j/\gamma_j = 1 - \beta_j/\gamma_j - r\sum_i \sigma_{ij}a_i/\gamma_j = \sum_i \theta_{ji} - r\sum_i \theta_{ji}(1-\rho\eta_i)$$

$$\implies \boldsymbol{\eta} = \delta r\boldsymbol{\Theta}\mathbf{1} + r\boldsymbol{\Theta}\boldsymbol{\eta} = \delta(\boldsymbol{I} - r\boldsymbol{\Theta})^{-1} r\boldsymbol{\Theta}\mathbf{1} = \delta\left[\sum_{s=1}^{\infty}\left(\frac{1}{1+\rho\delta}\boldsymbol{\Theta}\right)^s\right]\mathbf{1}$$

$$\iff \boldsymbol{\eta} = \frac{\delta}{1+\rho\delta}\boldsymbol{\Theta}\mathbf{1} + \frac{1}{1+\rho\delta}\boldsymbol{\Theta}\boldsymbol{\eta}.$$

A.6 Heterogeneous Adjustment Costs

Suppose adjustment cost is sector-pair-specific (δ_{ij}). Let $\boldsymbol{\Omega}$ denote the matrix whose ij-th entry is $\frac{\sigma_{ij}}{1+\rho\delta_{ij}}$. We now show the welfare impact $\boldsymbol{v}$ is

$$\boldsymbol{v}' \equiv \frac{1}{\rho}\boldsymbol{\beta}'\left[(\boldsymbol{I} - \boldsymbol{\Sigma})^{-1} - (\boldsymbol{I} - \boldsymbol{\Omega})^{-1}\right].$$

Following the proof to Lemma 1 in Appendix Section A.1, one can setup the Hamiltonian and find

$$\beta_j + \sum_i \delta_{ij}^{-1}\mu_{ij} = \left(\delta_{jk}^{-1} + \rho\right)\mu_{jk}/\sigma_{jk} \tag{32}$$

Let $\widetilde{\gamma}_j \equiv \left(\delta_{jk}^{-1}+\rho\right)\mu_{jk}/\sigma_{jk}$, the previous equation becomes

$$\beta_j + \sum_i \frac{1}{1+\rho\delta_{ij}}\widetilde{\gamma}_i\sigma_{ij} = \widetilde{\gamma}_j \iff \widetilde{\boldsymbol{\gamma}}' = \boldsymbol{\beta}'\left(\boldsymbol{I}-\boldsymbol{\Omega}\right)^{-1}$$

where $\boldsymbol{\Omega}_{ij} \equiv \frac{\sigma_{ij}}{1+\rho\delta_{ij}}$. Let $\boldsymbol{\delta}\circ\boldsymbol{\Omega}$ be the matrix whose ij-th entry is $\frac{\delta_{ij}\sigma_{ij}}{1+\rho\delta_{ij}}$. The welfare impact is

$$\begin{aligned} V\left(\widetilde{\boldsymbol{z}}\right) - V^{ss} &= -\sum_{ij}\mu_{ij}\left[\left(\boldsymbol{I}-\boldsymbol{\Sigma}\right)^{-1}\widetilde{\boldsymbol{z}}\right]_j \\ &= -\sum_{ij}\frac{\delta_{ij}}{1+\rho\delta_{ij}}\widetilde{\gamma}_i\sigma_{ij}\left[\left(\boldsymbol{I}-\boldsymbol{\Sigma}\right)^{-1}\widetilde{\boldsymbol{z}}\right]_j \\ &= \widetilde{\boldsymbol{\gamma}}'\left(\boldsymbol{\delta}\circ\boldsymbol{\Omega}\right)\left(\boldsymbol{I}-\boldsymbol{\Sigma}\right)^{-1}\widetilde{\boldsymbol{z}} \\ &= \boldsymbol{\beta}'\left(\boldsymbol{I}-\boldsymbol{\Omega}\right)^{-1}\left(\boldsymbol{\delta}\circ\boldsymbol{\Omega}\right)\left(\boldsymbol{I}-\boldsymbol{\Sigma}\right)^{-1}\widetilde{\boldsymbol{z}} \\ &= -\frac{1}{\rho}\boldsymbol{\beta}'\left[\left(\boldsymbol{I}-\boldsymbol{\Sigma}\right)^{-1}-\left(\boldsymbol{I}-\boldsymbol{\Omega}\right)^{-1}\right]\widetilde{\boldsymbol{z}}. \end{aligned}$$

A.7 General Production Functions and Adjustment Costs

Consider a steady-state of the economy. Let $\beta_j \equiv \partial\ln c\left(\{c_i\}_{i=1}^N\right)/\partial\ln c_j$ denote the steady-state consumption elasticity with respect to good j ($\boldsymbol{\beta}\equiv[\beta_j]$ is the corresponding vector), and $\sigma_{ij}\equiv\partial\ln q_i/\partial\ln m_{ij}$ is producer i's output elasticity with respect to input j in steady-state ($\boldsymbol{\Sigma}\equiv[\sigma_{ij}]$ is the corresponding matrix). Let $\omega_{ij}^{-1}\equiv\frac{\partial\ln g_{ij}}{\partial\ln s_{ij}}$ denote the rate at which purchased inputs s_{ij} expand the quantity of production inputs m_{ij}, again evaluated at the steady-state. Finally, let $\boldsymbol{\Omega}$ be the matrix whose ij-th entry is $\frac{\sigma_{ij}}{1+\rho\omega_{ij}}$. We now show that around the steady-state, the welfare impact of temporary, negative TFP shocks is

$$\left.\frac{\mathrm{d}V\left(\tilde{\boldsymbol{z}}\right)}{\mathrm{d}\widetilde{\boldsymbol{z}}}\right|_{\widetilde{\boldsymbol{z}}\equiv 0} = -\frac{1}{\rho}\left[\boldsymbol{\beta}'\left(\boldsymbol{I}-\boldsymbol{\Sigma}\right)^{-1}-\boldsymbol{\beta}'\left(\boldsymbol{I}-\boldsymbol{\Omega}\right)^{-1}\right].$$

We follow the proof strategy in Appendix Section A.1. Setup the planner's problem as

$$V\left(\{\ln m_{ij}(0)\}\right) = \max_{\{\ell_j(\cdot),v_{ij}(\cdot)\}}\int e^{-\rho t}\ln c\left(\left\{\ln f_j\left(\ell_j,\{m_{jk}\}\right)+\ln\left(1-\sum_i v_{ij}\right)\right\}_{j=1}^N\right)\mathrm{d}t$$

$$\text{s.t. } \frac{\mathrm{d}\ln m_{ij}(t)}{\mathrm{d}t} = g_{ij}\left(\ln v_{ij}+\ln f_j\left(\ell_j,\{\ln m_{jk}\}\right),\ln m_{ij}\right)$$

$$\sum_j \ell_j = \bar{\ell}$$

where recall $v_{ij} \equiv s_{ij}/q_j$ is the share of good j sent to producer i.[15] Let us form the current-value Hamiltonian:

$$\begin{aligned} H\left(\left\{\ell_j\right\}, \left\{\ln m_{jk}\right\}, \left\{v_{ij}\right\}, t\right) &= \ln c\left(\left\{\ln f_j\left(\ell_j, \left\{m_{jk}\right\}\right) + \ln\left(1 - \sum_i v_{ij}\right)\right\}_j\right) \\ &\quad + \sum_{ij} \mu_{ij} g_{ij}\left(\ln v_{ij} + \ln f_j\left(\ell_j, \left\{\ln m_{jk}\right\}\right), \ln m_{ij}\right) + \lambda\left[\bar{\ell} - \sum_j \ell_j\right]. \end{aligned}$$

Let $\xi_{ij}^{-1} \equiv \partial g_{ij}\left(\ln v_{ij} + \ln f_j\left(\ell_j, \left\{\ln m_{jk}\right\}\right), \ln m_{ij}\right)/\partial \ln m_{ij}$ denote the second partial derivate of g evaluated at the steady-state. Let $\alpha_j \equiv \frac{\partial \ln f_j}{\partial \ln \ell_j}$ denote the labor elasticities evaluated at the steady-state.

By the maximum principle,

$$H_{\ell_j} = 0 \iff \frac{\alpha_j\left(\beta_j + \sum_i \mu_{ij}\omega_{ij}^{-1}\right)}{\ell_j} = \lambda \quad \text{for all } j. \tag{33}$$

$$H_{v_{jk}} = 0 \iff \frac{\beta_j}{1 - \sum_i v_{ij}} = \frac{\mu_{ij}\omega_{ij}^{-1}}{v_{ij}} \tag{34}$$

$$H_{\ln m_{jk}} = \rho\mu_{jk} - \dot{\mu}_{jk} \iff \beta_j\sigma_{jk} + \mu_{jk}\xi_{jk}^{-1} + \sum_i \mu_{ij}\omega_{ij}^{-1}\sigma_{jk} = \rho\mu_{jk} - \dot{\mu}_{jk} \tag{35}$$

Evaluating the planner's problem at the steady-state $\left\{m_{ij}^*\right\}$, $\dot{\mu}_{ij} = 0$ for all i, j, and

$$g\left(\ln v_{ij} + \ln f\left(\ell_j, \left\{x_{jk}\right\}\right), x_{ij}\right) = 0.$$

Given that $s_{ij}^* = m_{ij}^*$ in a steady-state—hence $\ln s_{ij}^* = x_{ij}^*$—and by local-homogeneity and Euler's theorem,

$$\omega_{ij}^{-1}\ln s_{ij}^* + x_{ij}^*\xi_{ij}^{-1} = 0 \implies \omega_{ij} = -\xi_{ij} \quad \text{for all } i, j.$$

Equation (35) implies

$$\beta_j + \sum_i \mu_{ij}\omega_{ij}^{-1} = \frac{\mu_{jk}}{\sigma_{jk}}\left(\rho + \omega_{jk}^{-1}\right),$$

which coincides with (32) and we can simply interpret ω_{ij} as the local adjustment costs. Specifically, let $\boldsymbol{\Omega}$ denote the matrix whose ij-th entry is $\frac{\sigma_{ij}}{1+\rho\omega_{ij}}$, then around the steady-state, the non-parametric welfare elasticity to temporary shocks is

$$\left.\frac{\mathrm{d}V\left(\widetilde{\boldsymbol{z}}\right) - V^{ss}}{\mathrm{d}\widetilde{\boldsymbol{z}}}\right|_{\widetilde{\boldsymbol{z}}\equiv\boldsymbol{0}} = -\frac{1}{\rho}\boldsymbol{\beta}'\left[\left(\boldsymbol{I} - \boldsymbol{\Sigma}\right)^{-1} - \left(\boldsymbol{I} - \boldsymbol{\Omega}\right)^{-1}\right], \tag{36}$$

[15] Note that writing the law of motion in logs is without loss of generality; the only requirement on g is that it is locally homogeneous of degree one when the two arguments are equal; the property holds in logs if and only if it also holds in levels.

which follows the same logic as in Appendix Section A.6.

A.8 Welfare Impact under Two-Sided Adjustment Costs

We now derive the impact of a temporary, negative TFP shock $\tilde{\boldsymbol{z}}$ that affects the economy during $t \in [0, T)$ on welfare, assuming that both the contraction and expansion of inputs must be gradual. Integrating the law of motion for $\boldsymbol{x}(t)$, we get

$$\boldsymbol{x}(t) = \begin{cases} -e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})t}\tilde{\boldsymbol{z}} & t < T \\ e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})(t-T)}\left(\boldsymbol{I} - e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})T}\right)\tilde{\boldsymbol{z}} & t \geq T \end{cases}$$

For $t < T$,

$$\begin{aligned} \ln \boldsymbol{q}(t) &= \ln q(0) + \delta^{-1}\boldsymbol{\Sigma}\int_0^t \boldsymbol{x}(s)\, \mathrm{d}s \\ &= \ln q^{ss} - \tilde{\boldsymbol{z}} - \delta^{-1}\boldsymbol{\Sigma}\int_0^t e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})s}\, \mathrm{d}s\tilde{\boldsymbol{z}} \\ &= \ln q^{ss} - \tilde{\boldsymbol{z}} - \boldsymbol{\Sigma}(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\left[\boldsymbol{I} - e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})t}\right]\tilde{\boldsymbol{z}} \\ &= \ln q^{ss} - (\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\tilde{\boldsymbol{z}} + \boldsymbol{\Sigma}(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})t}\tilde{\boldsymbol{z}} \end{aligned}$$

$$\ln q(T) = \ln q^{ss} - \boldsymbol{\Sigma}(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\tilde{\boldsymbol{z}} + \boldsymbol{\Sigma}(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})T}\tilde{\boldsymbol{z}}$$

for $t > T$,

$$\begin{aligned} \ln q(t) &= \ln q(T) + \delta^{-1}\boldsymbol{\Sigma}\int_0^{t-T}\boldsymbol{x}(T+s)\, \mathrm{d}s \\ &= \ln q(T) + \delta^{-1}\boldsymbol{\Sigma}\left(\int_0^{t-T} e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})s}\, \mathrm{d}s\right)x(T) \\ &= \ln q(T) + \boldsymbol{\Sigma}(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\left(\boldsymbol{I} - e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})(t-T)}\right)x(T) \\ &= \ln q^{ss} - \boldsymbol{\Sigma}(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\tilde{\boldsymbol{z}} + \boldsymbol{\Sigma}(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})T}\tilde{\boldsymbol{z}} \\ &\quad + \boldsymbol{\Sigma}(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\left(\boldsymbol{I} - e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})(t-T)}\right)\left(\boldsymbol{I} - e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})T}\right)\tilde{\boldsymbol{z}} \\ &= \ln q^{ss} - \boldsymbol{\Sigma}(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})(t-T)}\left(e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})T} - \boldsymbol{I}\right)\tilde{\boldsymbol{z}} \end{aligned}$$

Welfare:

$$
\begin{aligned}
& V(\tilde{\boldsymbol{z}};T) - V^{ss} \\
= \; & \int_0^T e^{-\rho s}\left(\ln c(s) - \ln c_0^{ss}\right)\mathrm{d}s + e^{-\rho T}\int_0^\infty e^{-\rho s}\left(\ln c(T+s) - \ln c_0^{ss}\right)\mathrm{d}s \\
= \; & \boldsymbol{\beta}'\Big\{\int_0^T e^{-\rho s}\left(-(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1} + \boldsymbol{\Sigma}(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})s}\right)\mathrm{d}s \\
& + e^{-\rho T}\int_0^\infty e^{-\rho s}\left(-\boldsymbol{\Sigma}(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})s}\left(e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})T} - \boldsymbol{I}\right)\right)\mathrm{d}s\Big\}\widetilde{\boldsymbol{z}} \\
= \; & -\boldsymbol{\beta}'(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\Big\{\int_0^T e^{-\rho s}\boldsymbol{I} - \boldsymbol{\Sigma}e^{-\delta^{-1}((1+\rho\delta)\boldsymbol{I}-\boldsymbol{\Sigma})s}\,\mathrm{d}s \\
& - \boldsymbol{\Sigma}\left(e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})T} - \boldsymbol{I}\right)e^{-\rho T}\int_0^\infty\left(e^{-\delta^{-1}((1+\rho\delta)\boldsymbol{I}-\boldsymbol{\Sigma})s}\right)\mathrm{d}s\Big\}\widetilde{\boldsymbol{z}} \\
= \; & -\boldsymbol{\beta}'(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\Big\{\frac{1}{\rho}\left(1-e^{-\rho T}\right) - \boldsymbol{\Sigma}\delta\left((1+\rho\delta)\boldsymbol{I}-\boldsymbol{\Sigma}\right)^{-1}\left(\boldsymbol{I} - e^{-\delta^{-1}((1+\rho\delta)\boldsymbol{I}-\boldsymbol{\Sigma})T}\right) \\
& - \boldsymbol{\Sigma}\left(e^{-\delta^{-1}((1+\rho\delta)\boldsymbol{I}-\boldsymbol{\Sigma})T} - e^{-\rho T}\right)\delta\left((1+\rho\delta)\boldsymbol{I}-\boldsymbol{\Sigma}\right)^{-1}\Big\}\widetilde{\boldsymbol{z}} \\
= \; & -\frac{1}{\rho}\left(1-e^{-\rho T}\right)\boldsymbol{\beta}'(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\left(\boldsymbol{I} - \frac{\boldsymbol{\Sigma}\rho\delta}{1+\rho\delta}\left(\boldsymbol{I} - \frac{\boldsymbol{\Sigma}}{1+\rho\delta}\right)^{-1}\right)\widetilde{\boldsymbol{z}} \\
= \; & -\frac{1}{\rho}\left(1-e^{-\rho T}\right)\boldsymbol{\beta}'\left(\boldsymbol{I} - \frac{\boldsymbol{\Sigma}}{1+\rho\delta}\right)^{-1}\widetilde{\boldsymbol{z}}
\end{aligned}
$$

A.9 Output and Welfare Under Persistent TFP Shocks

Under exponential recovery of TFP,

$$\ln \boldsymbol{z}_t - \ln \boldsymbol{z}_0 = \left(1 - e^{-\phi t}\right)\widetilde{\boldsymbol{z}}$$

At time zero, the sectoral output follows

$$\ln \boldsymbol{q}(0) - \ln \boldsymbol{q}^{ss} = -(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\widetilde{\boldsymbol{z}}$$

As in the baseline model, we can define $x_j(t) \equiv \ln\sum_i s_{ij}(t) - \ln\sum_i m_{ij}(t)$, and, following the same strategy as in the proofs of Lemmas 1 and 2, we can show $\delta^{-1}x_j(t) = \frac{\mathrm{d}\ln m_{ij}}{\mathrm{d}t}$.

After time 0, sectoral output evolves according to

$$
\begin{aligned}
\frac{\mathrm{d}\ln q_i}{\mathrm{d}t} &= \frac{\mathrm{d}\ln z_i}{\mathrm{d}t} + \sum_j \sigma_{ij}\frac{\mathrm{d}\ln m_{ij}}{\mathrm{d}t} \\
\frac{\mathrm{d}\ln \boldsymbol{q}}{\mathrm{d}t} &= \frac{\mathrm{d}\ln \boldsymbol{z}}{\mathrm{d}t} + \delta^{-1}\boldsymbol{\Sigma}\boldsymbol{x}. \qquad (37)
\end{aligned}
$$

$\boldsymbol{x}$ evolves according to

$$\begin{aligned}\dot{\boldsymbol{x}} &= \frac{\mathrm{d}\ln\boldsymbol{q}}{\mathrm{d}t} - \delta^{-1}\boldsymbol{x} \\ &= \phi e^{-\phi t}\widetilde{\boldsymbol{z}} - \delta^{-1}\left(\boldsymbol{I} - \boldsymbol{\Sigma}\right)\boldsymbol{x} \qquad (38)\end{aligned}$$

with the initial condition that $\boldsymbol{x}(0) = \boldsymbol{0}$.

We can now derive the output path by first integrating (38) to obtain an explicit solution for $\boldsymbol{x}$, and then substitute into (37) to obtain an explicit solution for $\boldsymbol{q}$. We can then integrate to obtain the explicit solution for the welfare impact.

To do so, we eigendecompose $\boldsymbol{\Sigma} = \boldsymbol{U\Lambda W}$ where $\boldsymbol{\Lambda}$ is a diagonal matrix of eigenvalues $\{\lambda_k\}_{k=1}^N$ arranged in decreasing order by absolute values, and $\boldsymbol{W} = \boldsymbol{U}^{-1}$. We know that for each eigenvector $\boldsymbol{u}_h$, $\boldsymbol{\Sigma}\boldsymbol{u}_h = \lambda_h\boldsymbol{u}_h$.

We now write the TFP shock vector $\tilde{\boldsymbol{z}}$ as a linear combination $\{a_k\}_{k=1}^N$ of the right-eigenvectors $\tilde{\boldsymbol{z}} = \sum_{k=1}^N a_k\boldsymbol{u}_k$, where the weights can be recovered as $\boldsymbol{a} = \boldsymbol{W}\widetilde{\boldsymbol{z}}$. We consider each eigencomponent $\boldsymbol{u}_h$ separately, i.e., assume $\widetilde{\boldsymbol{z}} = \boldsymbol{u}_k$. There are two cases to consider.

Case 1. $\phi \neq \delta^{-1}(1-\lambda_k)$**.** Integrating (38) we get

$$\boldsymbol{x}_t = \frac{\phi}{\delta^{-1}(1-\lambda_k) - \phi}\left[e^{-\phi t} - e^{-\delta^{-1}(1-\lambda_k)t}\right]\boldsymbol{u}_k$$

note $\frac{\phi\delta}{1-\delta\phi-\lambda_k}\left(\frac{1}{\phi} - \frac{1}{\delta^{-1}(1-\lambda_k)}\right) = \frac{1}{\delta^{-1}(1-\lambda_k)}$, so

$$\begin{aligned}\int_0^t \boldsymbol{x}_s\,\mathrm{d}s &= \frac{\phi}{\delta^{-1}(1-\lambda_k)-\phi}\boldsymbol{u}_k\int_0^t\left[e^{-\phi s} - e^{-\delta^{-1}(1-\lambda_k)s}\right]\mathrm{d}s \\ &= \frac{\phi}{\delta^{-1}(1-\lambda_k)-\phi}\boldsymbol{u}_k\left[\frac{1}{\phi}\left(1-e^{-\phi t}\right) + \frac{1}{\delta^{-1}(1-\lambda_k)}\left[e^{-\delta^{-1}(1-\lambda_k)t} - 1\right]\right] \\ &= \frac{1}{\delta^{-1}(1-\lambda_k)}\boldsymbol{u}_k + \frac{\phi}{\delta^{-1}(1-\lambda_k)-\phi}\boldsymbol{u}_k\left[\frac{1}{\delta^{-1}(1-\lambda_k)}e^{-\delta^{-1}(1-\lambda_k)t} - \frac{1}{\phi}e^{-\phi t}\right]\end{aligned}$$

We integrate (37) to get

$$\begin{aligned}\ln\boldsymbol{q}(t) - \ln\boldsymbol{q}^{ss} &= \left(1-e^{-\phi t}\right)\boldsymbol{u}_k - (I-\Sigma)^{-1}\boldsymbol{u}_k + \delta^{-1}\boldsymbol{\Sigma}\int_0^t \boldsymbol{x}_s\,\mathrm{d}s \\ &= \frac{1}{(1-\lambda_k)-\phi\delta}\frac{\lambda_k\delta\phi}{(1-\lambda_k)}e^{-\delta^{-1}(1-\lambda_k)t}\boldsymbol{u}_k - \frac{1-\phi\delta}{(1-\lambda_k)-\phi\delta}e^{-\phi t}\boldsymbol{u}_k \qquad (39)\end{aligned}$$

Hence

$$
\begin{aligned}
&\int_0^\infty e^{-\rho t}\left(\ln \boldsymbol{q}(t)-\ln \boldsymbol{q}^{ss}\right)\,\mathrm{d}t \\
= &\left[\frac{\phi\delta}{(1-\lambda_k)-\phi\delta}\frac{\lambda_k}{(1-\lambda_k)}\frac{\delta}{\rho\delta+(1-\lambda_k)}-\frac{1}{(1-\lambda_k)-\phi\delta}\frac{1-\phi\delta}{\rho+\phi}\right]\boldsymbol{u}_k \\
= &\left[\frac{\phi\delta}{(1-\lambda_k)-\phi\delta}\frac{1}{\rho}\left(\frac{1}{1-\lambda_k}-\frac{1+\rho\delta}{1+\rho\delta-\lambda_k}\right)-\frac{1}{(1-\lambda_k)-\phi\delta}\frac{1-\phi\delta}{\rho+\phi}\right]\boldsymbol{u}_k \qquad (40)
\end{aligned}
$$

Case 2. $\phi=\delta^{-1}(1-\lambda_k)$. Integrating $\dot{x}_t$ we get

$$\boldsymbol{x}_t = t\phi e^{-\phi t}\boldsymbol{u}_k$$

Note $\phi\frac{\mathrm{d}\phi^{-1}e^{-\phi t}}{\mathrm{d}\phi}=-\left(\phi^{-1}+t\right)e^{-\phi t}$, so

$$
\begin{aligned}
\int_0^t \boldsymbol{x}_s\,\mathrm{d}s &= \frac{1}{\phi}\boldsymbol{u}_k+\lim_{x\to 0}\frac{\phi\boldsymbol{u}_k}{x}\left[\frac{1}{\phi+x}e^{-(\phi+x)t}-\frac{1}{\phi}e^{-\phi t}\right] \\
&= \frac{1}{\phi}\boldsymbol{u}_k+\phi\boldsymbol{u}_k\frac{\mathrm{d}\phi^{-1}e^{-\phi t}}{\mathrm{d}\phi} \\
&= \frac{1}{\phi}\boldsymbol{u}_k-\left(\phi^{-1}+t\right)e^{-\phi t}\boldsymbol{u}_k
\end{aligned}
$$

We integrate (37) to get

$$
\begin{aligned}
\ln \boldsymbol{q}(t)-\ln \boldsymbol{q}^{ss} &= \left(1-e^{-\phi t}\right)\boldsymbol{u}_k-(I-\Sigma)^{-1}\boldsymbol{u}_k+\delta^{-1}\boldsymbol{\Sigma}\int_0^t \boldsymbol{x}_s\,\mathrm{d}s \\
&= -\left(\frac{1}{1-\lambda_k}+\frac{t}{\delta}\lambda_k\right)e^{-\phi t}\boldsymbol{u}_k
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_0^\infty e^{-\rho t}\left(\ln \boldsymbol{q}(t)-\ln \boldsymbol{q}^{ss}\right)\,\mathrm{d}t &= -\int_0^\infty\left(\frac{1}{1-\lambda_k}+\frac{t}{\delta}\lambda_k\right)e^{-(\rho+\phi)t}\,\mathrm{d}t\boldsymbol{u}_k \\
&= -\frac{1}{\rho+\phi}\left(\frac{1}{1-\lambda_k}+\frac{\lambda_k}{\delta(\rho+\phi)}\right)\boldsymbol{u}_k
\end{aligned}
$$

Taking Stock. When $[(1-\phi\delta)\boldsymbol{I}-\boldsymbol{\Sigma}]$ is invertible; then (39) implies that the path of sectoral output following a generic TFP shock $\widetilde{\boldsymbol{z}}$ is

$$\ln \boldsymbol{q}(t)=\ln \boldsymbol{q}^{ss}+((1-\phi\delta)\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\left[\boldsymbol{\Sigma}\phi\delta(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}e^{-\delta^{-1}(\boldsymbol{I}-\boldsymbol{\Sigma})t}-(1-\phi\delta)e^{-\phi t}\right]\widetilde{\boldsymbol{z}}.$$

Also equation (40) implies the welfare impact is

$$V(\tilde{\boldsymbol{z}};\phi)-V^{ss}=\boldsymbol{\beta}'((1-\phi\delta)\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\left[\frac{\phi\delta}{\rho}\left((\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}-\left(\boldsymbol{I}-\frac{\boldsymbol{\Sigma}}{1+\rho\delta}\right)^{-1}\right)-\frac{1-\phi\delta}{\rho+\phi}\boldsymbol{I}\right]\widetilde{\boldsymbol{z}}.$$

When $[(1-\phi\delta)\boldsymbol{I}-\boldsymbol{\Sigma}]$ is not invertible, we write the TFP shock vector $\widetilde{\boldsymbol{z}}$ as a linear combination $\{a_k\}_{k=1}^{N}$ of the right-eigenvectors $\tilde{\boldsymbol{z}} = \sum_{k=1}^{N} a_k \boldsymbol{u}_k$, where the weights can be recovered as $\boldsymbol{a} = \boldsymbol{W}\tilde{\boldsymbol{z}}$. Define

$$\zeta_k(t) \equiv \begin{cases} \frac{1}{(1-\lambda_k)-\phi\delta}\frac{\lambda_k\delta\phi}{(1-\lambda_k)} e^{-\delta^{-1}(1-\lambda_k)t} - \frac{1-\phi\delta}{(1-\lambda_k)-\phi\delta} e^{-\phi t} & \text{if } \phi \neq \delta^{-1}(1-\lambda_k) \\ -\left(\frac{1}{1-\lambda_k} + \frac{t}{\delta}\lambda_k\right) e^{-\phi t} & \text{if } \phi = \delta^{-1}(1-\lambda_k) \end{cases}$$

$$\mu_k \equiv \begin{cases} \frac{\phi\delta}{(1-\lambda_k)-\phi\delta}\frac{1}{\rho}\left(\frac{1}{1-\lambda_k} - \frac{1+\rho\delta}{1+\rho\delta-\lambda_k}\right) - \frac{1}{(1-\lambda_k)-\phi\delta}\frac{1-\phi\delta}{\rho+\phi} & \text{if } \phi \neq \delta^{-1}(1-\lambda_k) \\ -\frac{1}{\rho+\phi}\left(\frac{1}{1-\lambda_k} + \frac{\lambda_k}{\delta(\rho+\phi)}\right) & \text{if } \phi = \delta^{-1}(1-\lambda_k) \end{cases}$$

We have

$$\ln \boldsymbol{q}(t) = \ln \boldsymbol{q}^{ss} + \sum_k \zeta_k(t)\, a_k \boldsymbol{u}_k$$

$$V(\tilde{\boldsymbol{z}};\phi) - V^{ss} = \sum_k \mu_k a_k \boldsymbol{u}_k.$$

A.10 Welfare Impact of Permanent Shocks

We derive the welfare impact of permanent shocks by starting with the welfare impact of persistent shocks and take taking the limit as $\phi \to 0$ (so that TFP never recovers). As $\phi \to 0$, $[(1-\phi\delta)\boldsymbol{I} - \boldsymbol{\Sigma}]$ is invertible, and

$$\lim_{\phi\to 0}\left(V(\tilde{\boldsymbol{z}};\phi) - V^{ss}\right) = \frac{1}{\rho}\boldsymbol{\beta}'(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}\tilde{\boldsymbol{z}}.$$

Since $V^{ss} = \frac{1}{\rho}\ln c^{ss}$, the Domar weight $\boldsymbol{\gamma}' \equiv \boldsymbol{\beta}'(\boldsymbol{I}-\boldsymbol{\Sigma})^{-1}$ thus characterizes the cross steady-state consumption differences resulting from permanent TFP shocks.

A.11 Eigendecomposition

Consider the Domar weight

$$\begin{aligned} \boldsymbol{\gamma}' &= \boldsymbol{\beta}'\left(\sum_{s=0}^{\infty}\boldsymbol{\Sigma}^s\right) = \boldsymbol{\beta}'\boldsymbol{U}\left(\sum_{s=0}^{\infty}\boldsymbol{\Lambda}^s\right)\boldsymbol{W} \\ &= \boldsymbol{\beta}'\sum_{k=1}^{N}\left(\sum_{s=0}^{\infty}\lambda_k^s\right)\boldsymbol{u}_k\boldsymbol{w}_k' = \boldsymbol{\beta}'\sum_{k=1}^{N}\frac{1}{1-\lambda_k}\boldsymbol{u}_k\boldsymbol{w}_k'. \end{aligned}$$

The welfare impact

$$\begin{aligned}
\boldsymbol{v}' &= \frac{1}{\rho}\boldsymbol{\beta}'\sum_{s=0}^{\infty}\left(1-(1+\rho\delta)^{-s}\right)\boldsymbol{\Sigma}^s \\
&= \frac{1}{\rho}\boldsymbol{\beta}'\sum_{k=1}^{N}\left(\frac{1}{1-\lambda_k}-\frac{1}{1-\frac{1}{1+\rho\delta}\lambda_k}\right)\boldsymbol{u}_k\boldsymbol{w}_k' \\
&= \delta\boldsymbol{\beta}'\sum_{k=1}^{N}\frac{\lambda_k}{(1-\lambda_k)(1+\rho\delta-\lambda_k)}\boldsymbol{u}_k\boldsymbol{w}_k'.
\end{aligned}$$

A.12 Connection to Time-to-Build and Long and Plosser (1983)

We show our law of motion (8) for intermediate inputs, microfounded by exponential adjustment costs, is to first-order equivalent to a continuous-time formulation of the time-to-build specification in Long and Plosser (1983).

Specifically, suppose there are no adjustment costs but instead, after each intermediate input j is produced, it must go through logistical delays before it can arrive at the production lines of input-using sector i. In Long and Plosser (1983)'s discrete-time formulation, goods arrive with one period delay. Since our model is in continuous-time, we assume intermediate inputs arrive from sellers to buyers following a Poisson process with rate δ^{-1}, corresponding to an exponentially distributed delay with mean δ. Formally, let $a_{ij}(t)$ denote the stock of good j sold to but have not arrived at sector i by time t. The law of motion for a_{ij} is

$$\dot{a}_{ij} = s_{ij} - m_{ij}, \tag{41}$$

which states that the rate of change in the stock of good j on its way to sector i is the difference between the quantity of new purchase (s_{ij}) and the quantity of arrival (m_{ij}). Given that goods arrive with Poisson rate δ^{-1}, the quantity of arrival follows $m_{ij}(t) = \delta^{-1}a_{ij}(t)$, and, combining with equation (41), we derive the law of motion for the use of intermediate inputs:

$$\dot{m}_{ij} = \delta^{-1}(s_{ij} - m_{ij}). \tag{42}$$

Under this microfoundation, the parameter δ can also be interpreted as the backlog ratio: it measures the average delay between when inputs are ordered and delivered.

Under the time-to-build formulation, the law of motion (42) states that the rate of change in m_{ij} is linear in the difference between new purchase orders s_{ij} and quantity delivered m_{ij}. By contrast, the law of motion (8) in our baseline adjustment cost formulation states that the growth rate in m_{ij} is linear in the log-difference $(\ln s_{ij} - \ln m_{ij})$. The two formulations are equivalent to first-order; that is, when s_{ij}/m_{ij} is close to one, equation (42) can be re-written as

$$\frac{\dot{m}_{ij}}{m_{ij}} = \delta^{-1}\left(\frac{s_{ij} - m_{ij}}{m_{ij}}\right) \approx \delta^{-1}\left(\ln s_{ij} - \ln m_{ij}\right).$$

Hence, when TFP shocks are small—so that allocations at time 0 are not too far from the eventual steady-state—our model predictions on the path of sectoral output closely matches the predictions of a dynamic network model with time-to-build. The main advantage of our formulation is tractability. As we have shown, a log-linear law of motion (8) affords us closed-form solutions for the entire path of sectoral output, thereby enabling us to derive substantive analytic insights of how the network structure affects the economy's susceptibility to and recovery after temporary shocks.

B Factor Structure of the U.S. Input-Output Table: Additional Empirical Results

Figure 7. Welfare impact from the first h eigencomponents ($\boldsymbol{v}_{(h)}$) plotted against $\boldsymbol{v}$

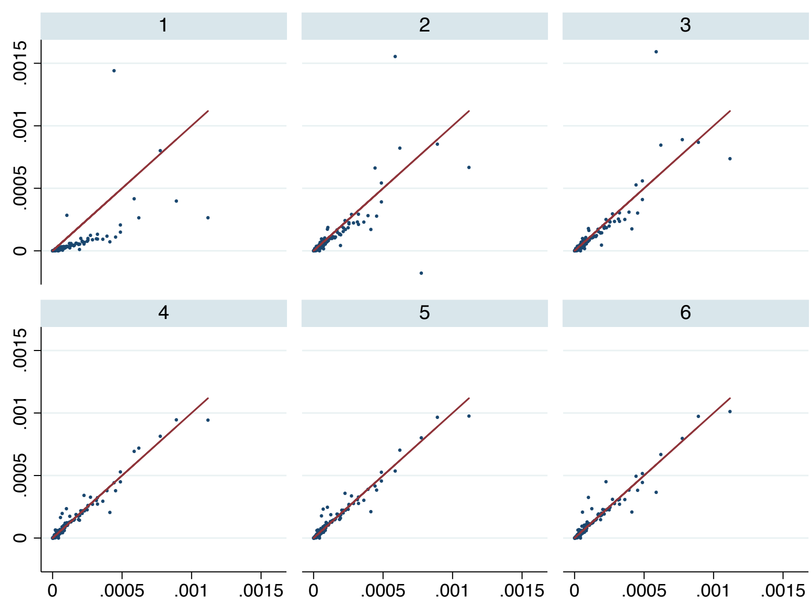


Figure 8. Domar weights from the first h eigencomponents ($\boldsymbol{\gamma}_{(h)}$) plotted against $\boldsymbol{\gamma}$

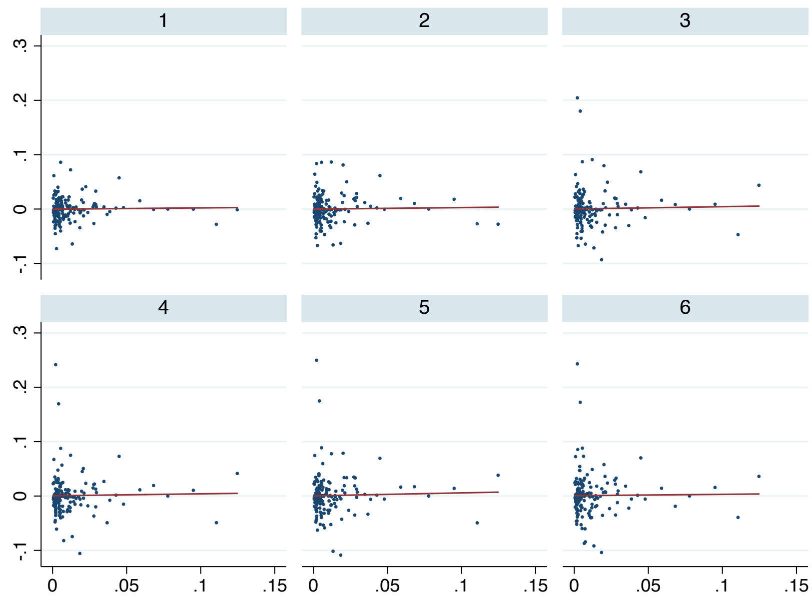
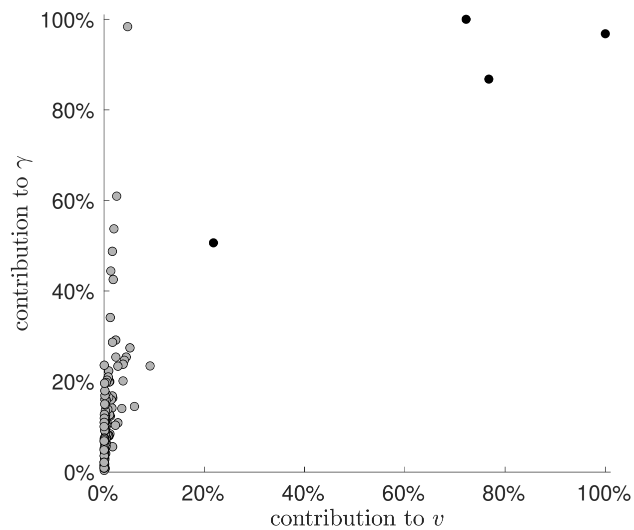


Table 6. The 1st & 2nd eigenvectors: 10 largest entries by absolute value

$[\boldsymbol{u}_1]_i$		$[\boldsymbol{u}_2]_i$	
Nonferrous metal (except aluminum) production and processing	0.439	Agencies, brokerages, and other insurance related activities	0.574
Alumina and aluminum production and processing	0.221	Insurance carriers	0.334
Other electrical equipment and component manufacturing	0.213	Animal slaughtering and processing	0.128
Railroad rolling stock manufacturing	0.187	Dairy product manufacturing	0.114
Motor vehicle manufacturing	0.178	Electrical equipment manufacturing	-0.097
Steel product manufacturing from purchased steel	0.178	Steel product manufacturing from purchased steel	-0.103
Forging and stamping	0.175	Forging and stamping	-0.110
Boiler, tank, and shipping container manufacturing	0.163	Alumina and aluminum production and processing	-0.140
Iron and steel mills and ferroalloy manufacturing	0.159	Other electrical equipment and component manufacturing	-0.176
Motor vehicle parts manufacturing	0.153	Nonferrous metal (except aluminum) production and processing	-0.416

Figure 9. The contribution of each eigencomponent to $\boldsymbol{\gamma}$ (Domar weight) and $\boldsymbol{v}$ (welfare impact of temporary shocks) relative to the first component

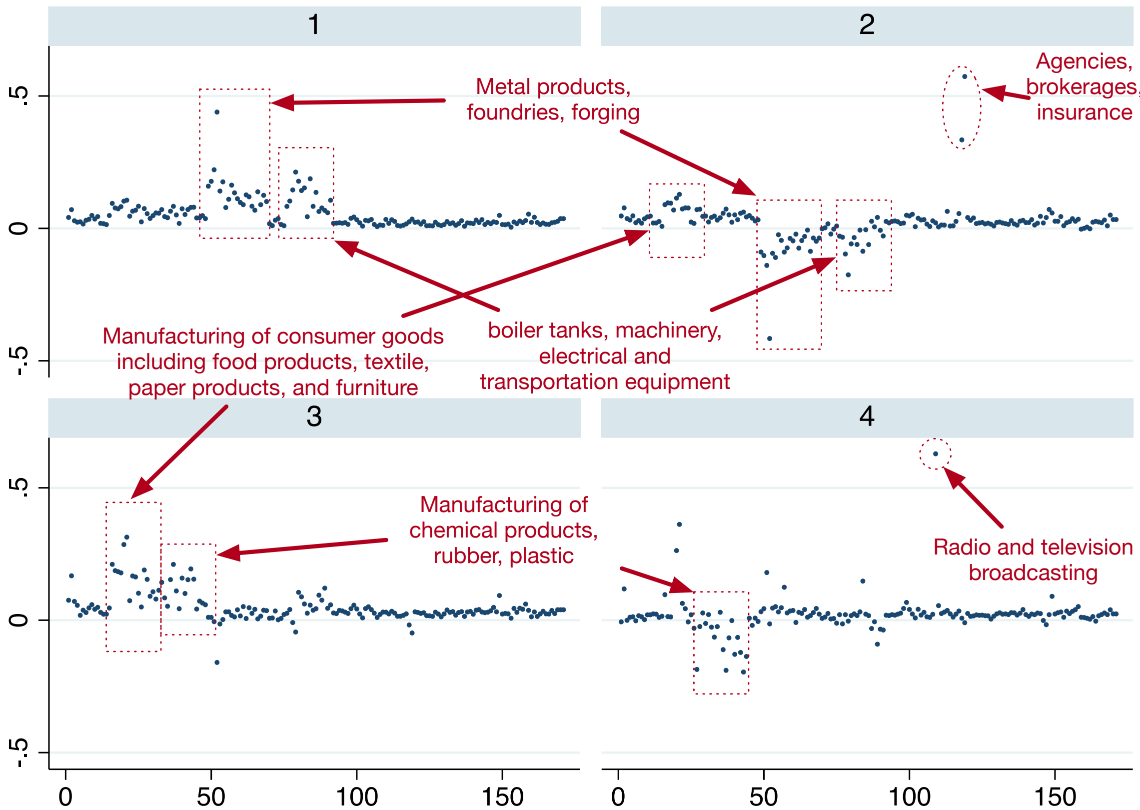


Notes. The figure contrasts the contribution of each eigencomponent to the Domar weight $\boldsymbol{\gamma}$—as determinants of the welfare response to permanent shocks—and to $\boldsymbol{v}$—as determinants of the welfare response to temporary shocks. Each point in the figure represents an eigencomponent $k \in \{1, \ldots, 171\}$. The X-coordinate is $|\boldsymbol{v}'\boldsymbol{u}_k|$, the k-th eigencomponent's contribution to $\boldsymbol{v}$. The Y-coordinate is $|\boldsymbol{\gamma}'\boldsymbol{u}_k|$, the k-th eigencomponent's contribution to $\boldsymbol{\gamma}$. For the two axes to be comparable, we normalize the contribution of the most important eigencomponent to 100% on each axis. The figure shows that there are only four eigencomponents (black circles) with X-coordinates above 10%, meaning the fifth most important eigencomponent contributes to less than 10% as the most important component for the welfare impact $\boldsymbol{v}$ of temporary shocks. By contrast, for permanent shocks, a large number (101 out of 171) of eigencomponents have their contributions exceeding 10% of the most important component.

Table 7. The 3rd & 4th eigenvectors: 10 largest entries by absolute value

$[\boldsymbol{u}_3]_i$		$[\boldsymbol{u}_4]_i$	
Animal slaughtering and processing	0.314	Radio and television broadcasting	0.628
Dairy product manufacturing	0.286	Animal slaughtering and processing	0.362
Animal food manufacturing	0.211	Dairy product manufacturing	0.263
Resin, synthetic rubber, and artificial synthetic fibers and filaments manufacturing	0.211	Alumina and aluminum production and processing	0.180
Plastics product manufacturing	0.194	Railroad rolling stock manufacturing	0.148
Textile mills and textile product mills	0.190	Paint, coating, and adhesive manufacturing	-0.129
Grain and oilseed milling	0.187	Rubber product manufacturing	-0.137
Sugar and confectionery product manufacturing	0.183	Textile mills and textile product mills	-0.186
Fruit and vegetable preserving and specialty food manufacturing	0.179	Resin, synthetic rubber, and artificial synthetic fibers and filaments manufacturing	-0.189
Animal production and aquaculture	0.168	Plastics product manufacturing	-0.196

Figure 10. The first four eigenvectors of $\mathbf{\Sigma}$



Notes. The X-axis represent the sectoral ordering according to the BLS input-output table, which roughly arranges broad sector groups by agriculture, food manufacturing, chemical products, metals, heavy manufacturing, and services. In the figure, we indicate the broad groups of sectors that these eigenvectors represent; Appendix Tables 6 and 7 provide more detailed lists of sector names.

Table 8. Low dimensional representation of TFP shocks to vulnerable sectors in the U.S.

10 sectors with the highest v_i	Loadings on the first 4 eigenvectors			
	1st	2nd	3rd	4th
Real estate	0.29	0.10	0.62	1.05
Wholesale trade	0.44	0.02	0.70	0.39
Agencies, brokerages, and other insurance related activities	0.89	1.54	-1.50	-0.38
Oil and gas extraction	0.29	0.03	0.86	-0.64
Basic chemical manufacturing	0.46	0.06	1.75	-4.57
Management of companies and enterprises	0.17	0.03	0.37	0.20
Petroleum and coal products manufacturing	0.23	0.02	0.51	-0.15
Advertising, public relations, and related services	0.12	0.04	0.26	0.39
Nonferrous metal (except aluminum) production & processing	1.60	-0.20	-1.19	-0.43
Motor vehicle parts manufacturing	0.08	0.01	0.15	0.15